A MINIMUM PROBLEM WITH FREE BOUNDARY FOR A POSSIBLY DEGENERATE (OR SINGULAR) INHOMOGENEOUS QUASILINEAR ELLIPTIC OPERATOR

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We take Ω a smooth bounded domain in $\mathbb{R}^N$ and we consider the problem of minimizing the functional,

$$\mathcal{J}(u) = \int_\Omega G(|\nabla u|) + \lambda \chi_{\{u>0\}}\, dx,$$

in the class of functions

$$\mathcal{K} = \left\{ v \in L^1(\Omega) : \int_\Omega G(|\nabla u|)\, dx < \infty,\ v = \varphi_0 \right\},$$

for a given nonnegative $\varphi_0 \in L^1(\Omega)$ such that $\int_\Omega G(|\nabla \varphi_0|)\, dx < \infty$.

The conditions on the function G imply that G is (not necessarily strictly) convex and they allow for a different behavior at 0 and at ∞. Moreover the set of functions that satisfy these conditions include inhomogeneous functions.

We show that a minimizer always exists and then, we prove the following properties for minimizers:

First we prove the Hölder continuity. With this result we show that minimizers are solutions of an elliptic differential equation $\mathcal{L}u = 0$ in their positivity sets. To prove the continuity we need to prove a Caccippoli type inequality for subsolutions of this equation. Then, we prove the uniform Lipschitz continuity (i.e. that $|\nabla u|$ is bounded in every compact subset of Ω containing a free boundary point, by a constant independent of the minimizer u).

We also have that the free boundary $\Omega \cap \partial\{u > 0\}$ has locally finite perimeter in Ω. As usual, we define the reduce boundary by $\partial_{red}\{u > 0\} := \{x \in \Omega \cap \partial\{u > 0\}/|\nu_u(x)| = 1\}$ where $\nu_u(x)$ is the outer normal derivative, when it exists, and $\nu_u(x) = 0$ otherwise and we prove that minimizers have an asymptotic development at points in their reduced free boundary.

So that, minimizers satisfy (in a weak sense) the following free boundary problem,

$$\begin{cases} \mathcal{L}u = 0 & \text{in} \quad \Omega \cap \{u > 0\}, \\ u = 0, \quad |\nabla u| = \lambda^* & \text{on} \quad \Omega \cap \partial\{u > 0\}. \end{cases} \tag{1}$$

where $\mathcal{L}u = \operatorname{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)$, $g(t) = G'(t)$, and λ^* is a constant that verifies $g(\lambda^*)\lambda^* - G(\lambda^*) = \lambda$.

These results lead to the consideration of weak solutions of the problem (1). We give two alternative definitions of weak solution and prove the $C^{1,\alpha}$ regularity of their free boundaries.