Trajectory and global attractors of the boundary value problem for motion equations of viscoelastic medium

Dmitry A. Vorotnikov, Victor G. Zvyagin

Department of Mathematics, Voronezh State University, 394006, Universitetskaya Pl.,1, Voronezh, Russia

Introduction

Attractors for systems of differential equations or for dynamical systems are the sets to which the solutions of an equation or trajectories of a system are eventually attracted (after damping of transient processes). As a rule, to the condition of attraction one adds the conditions of strict invariance, minimality and compactness. The classical examples of attractors are equilibrium points or fixed points of maps, limit cycles or tori surfaces for quasiperiodic motion.

In work [7] O.A. Ladyzenskaya proposed to investigate attractors as the basic object of the turbulence theory for evolution systems with dissipation and for equations of hydrodynamics, in particular. In that work she also proved existence and investigated properties of the global attractor for the two-dimensional Navier-Stokes system. The further study of this attractor was carried out by various authors (see [2, 13]). Research of the three-dimensional case was restricted by the difficulty that in this case existence of global in time strong solutions for the basic initial-boundary value problem and uniqueness of weak solutions of this problem are not proved till now. In this connection there was developed the theory of trajectory attractors (see e.g. [3, 9, 14]), which allows to construct a global attractor for various evolution equations without uniqueness of solutions of the corresponding Cauchy problem. In particular, there was proved existence of trajectory and global attractors for weak solutions of the three-dimensional Navier-Stokes system.

However, it turned out that this theory cannot be directly applied for the following boundary value problem for the autonomous motion equations of a viscoelastic medium with the Jeffreys constitutive law:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} + grad\ p = Div\ \sigma + f \tag{0.1}$$

$$\sigma + \lambda_1\left(\frac{\partial \sigma}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \sigma}{\partial x_i}\right) = 2\eta\left(\mathcal{E} + \lambda_2\left(\frac{\partial \mathcal{E}}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \mathcal{E}}{\partial x_i}\right)\right) \tag{0.2}$$

$$div\ u = 0 \tag{0.3}$$

$$u\Big|_{\partial\Omega} = 0 \tag{0.4}$$

Here Ω is an arbitrary bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$, u is an unknown vector of velocity of points of the medium, p is an unknown function of pressure, σ is an unknown deviator of the stress tensor (all of them depend on a point $x \in \Omega$ and a moment of time t), $f(x)$ is the body force, $\mathcal{E} = \mathcal{E}(u) = (\mathcal{E}_{ij})$, $\mathcal{E}_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the strain velocity tensor, $\eta > 0$ is the viscosity of the medium, λ_1 is the relaxation time, λ_2 is the retardation time, $0 < \lambda_2 < \lambda_1$. The gradient *grad* and the divergence *div* are taken with respect to the variable x. The divergence *Div* of a tensor is the vector with the coordinates $(Div\ \sigma)_j = \sum_{i=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_i}$. The density of the medium is considered to be equal to one.

Equation (0.1) is the general motion equation in the form of Cauchy [4]. Equation (0.2) is the Jeffreys constitutive law. This relation describes materials like solutions of polymers, bitumens, concrete, the earth's crust [8]. Equation (0.3) is the equation of continuity [4]. Equation (0.4) is the non-slip condition on the boundary of the domain Ω.

Regarding the attractors of non-Newtonian flows, we can mention paper [6] where the attractor of two-dimensional problem (0.1), (0.3), (0.4) with (0.2) substituted by the linearized constitutive relation was studied and paper [1] where the attractor of a regularized model of motion of a nonlinear-viscous fluid was constructed and investigated.

The difficulties in application of the theory of trajectory attractors to Jeffreys' problem (0.1)-(0.4) come from the following two points which are essential in this theory. These points are the assumption of invariance of the trajectory space of the equation for which attractors are constructed with respect to shifts along the time axis (or some generalization of this condition for the case of non-autonomous equations) and the requirement of constructing an attracting set which is contained in the trajectory space (it may be replaced [3] by the assumption that the family of trajectory spaces is closed in some special sense). But for Jeffreys' problem the classes of existence of solutions for the corresponding initial-boundary value problem (which are determined by a function space and an energy inequality) generate trajectory spaces which are either not invariant with respect to shifts along the time axis or too wide to show the existence of an attractor. The construction of an attracting set contained in the trajectory space is also a problem.

Existence of weak solutions for problem (0.1) - (0.4) for $n = 2, 3$ is proved in [15] using the approximating - topological method [16]. The problem of uniqueness of these solutions is open even at $n = 2$.

In the present work we introduce the concept of minimal trajectory attractor generalizing the concept of trajectory attractor from [14] for a trajectory space of an abstract evolution equation. With the help of this concept it is possible to construct the global attractor. It enables to obtain a number of theorems on existence of minimal trajectory and global attractors without assumptions of any invariance of the trajectory space of an equation. Theorems 2.1–2.3 do not require for the attracting set to be contained in the trajectory space. However, Theorems 2.4 and 2.5 show that under this assumption we can say more about the attractors. We also obtain some results about the properties and the structure of minimal trajectory attractors.

With the help of the obtained results we prove existence of minimal trajectory and global attractors for weak solutions of the boundary value problem (0.1) - (0.4) for autonomous motion equations of a viscoelastic medium at $n = 2, 3$.

1 Notations

We shall use the following notations. Most of them are standard.

Denote by $\mathbb{R}^{n\times n}$ the space of matrices of the order $n \times n$ with the following scalar product: for $A = (A_{ij})$, $B = (B_{ij})$

$$(A, B)_{\mathbb{R}^{n\times n}} = \sum_{i,j=1}^{n} A_{ij}B_{ij}$$

and by $\mathbb{R}^{n\times n}_S$ its subspace of symmetric matrices.

Denote by $\mathbb{R}^{n\times n\times n}$ the space of ordered collections of n matrices of the order $n \times n$ with the following scalar product: for $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$

$$(A, B)_{\mathbb{R}^{n\times n\times n}} = \sum_{i=1}^{n} (A_i, B_i)_{\mathbb{R}^{n\times n}}$$

The symbol ∇u stands for the Jacobi matrix of a vector function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. The symbol $\nabla\tau$ denotes the ordered collection of the Jacobi matrices of the columns of a matrix function $\tau : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n\times n}$.

The symbol $K(\cdot, ..., \cdot)$ stands for positive constants, depending continuously on arguments, which will be enumerated.

Below F stands for one of the spaces $\mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}_S$, $\mathbb{R}^{n\times n\times n}$.

We use the standard notations $L_p(\Omega, F)$, $W^\beta_p(\Omega, F)$, $H^\beta(\Omega, F) = W^\beta_2(\Omega, F)$ ($\beta \in \mathbb{R}$), $H^\beta_0(\Omega, F) = \overset{\circ}{W}{}^\beta_2(\Omega, F)$ ($\beta > 0$) for Lebesgue and Sobolev spaces of functions with values in F. Sometimes we shall write simply L_p instead of $L_p(\Omega, F)$ etc., if it is clear from the context which space F is used.

Parentheses denote the following bilinear form:

$$(u, v) = \int_\Omega (u(x), v(x))_F dx$$

The Euclid norm in F is denoted as $|\cdot|$ and in L_2 as $\|\cdot\|$. We shall also use the notation $\|v\|_1 = \|\nabla v\|$, $v \in H^1$.

By the symbol $C_0^\infty(\Omega, F)$ we denote the space of smooth functions with compact support in Ω and with values in F.

For brevity we denote by C_0^∞ the space $C_0^\infty(\Omega, \mathbb{R}_S^{n\times n})$.

Let

$\mathcal{V} = \{u \in C_0^\infty(\Omega, \mathbb{R}^n),\ div\ u = 0\}$.

Let the symbols H, V, V_δ ($\delta \in (0, 1]$) denote the closures of $\mathcal{V}$ in $L_2(\Omega, \mathbb{R}^n)$, $W_2^1(\Omega, \mathbb{R}^n)$, $W_2^\delta(\Omega, \mathbb{R}^n)$, respectively.

Following [12], we identify the space H and its conjugate space H^*. Therefore we have the embedding

$$V_\delta \subset H \equiv H^* \subset V_\delta^*$$

The value of a functional from H^{-m}, V^*, V_δ^* on an element from H_0^m, V, V_δ, respectively, will be denoted by brackets $\langle\cdot,\cdot\rangle$.

Since Ω is bounded, the inequality

$$\|v\| \leq K_0(\Omega)\|v\|_1, v \in V \tag{1.1}$$

is valid.

The symbols $C(J; X)$, $C_w(J; X)$, $L_2(J; X)$ etc. will denote the spaces of continuous, weakly continuous, quadratically integrable etc. functions on an interval $J \subset \mathbb{R}$ (which may be unbounded) with values in a Banach space X.

Let us remind that a pre-norm in the Frechet space $C([0, +\infty); X)$ may be defined by the formula

$$\|v\|_{C([0,+\infty);X)} = \sum_{i=0}^{+\infty} 2^{-i} \frac{\|v\|_{C([0,i];X)}}{1 + \|v\|_{C([0,i];X)}},$$

and in the space $C((-\infty, +\infty); X)$ by the formula

$$\|v\|_{C((-\infty,+\infty);X)} = \sum_{i=0}^{+\infty} 2^{-i} \frac{\|v\|_{C([-i,i];X)}}{1 + \|v\|_{C([-i,i];X)}}.$$

2 Attractors of abstract evolution equations

2.1 Basic definitions

Let E and E_0 be Banach spaces, $E \subset E_0$. Consider an abstract differential equation

$$u'(t) = A(u(t)), u(t) \in E \tag{2.1}$$

We shall investigate attractors of solutions of this equation which belong to the space $C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$.

It is supposed that the space E is reflexive. Then
$C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E) \subset C_w([0, +\infty); E)$ (see [12], chapter III, Lemma 1.4,

and also [11], p. 232 where it is shown that without the condition of reflexivity of E there may be no such embedding). Hence, the values of functions from $C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ belong to E at every $t\ge 0$.

Consider the translation (shift) operators $T(h), h\ge 0$,

$$T(h)(u)(t)=u(t+h),$$

$u\in C([0,+\infty);E_0), L_\infty(0,+\infty;E)$, $C((-\infty,+\infty);E_0)$ or $L_\infty(-\infty,+\infty;E)$.

For any fixed $h\ge 0$ the operators $T(h)$ are continuous bounded mappings of the spaces $C([0,+\infty);E_0)$ and $L_\infty(0,+\infty;E)$ into themselves.

Let some set

$$\mathcal{H}^+\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$$

of solutions (strong, weak, etc.) for equation (2.1) on the positive axis be fixed. The set $\mathcal{H}^+$ will be called trajectory space and its elements will be called trajectories.

Remarks. 1. We do **not** assume that $T(h)\mathcal{H}^+\subset\mathcal{H}^+$ for $h\ge 0$. 2. Below the concrete form of equation (2.1) is not significant but only presence of a trajectory space $\mathcal{H}^+$ is important and everything will depend only on the properties of this set. Generally speaking, the nature of $\mathcal{H}^+$ may be different from the one described above.

Definition 2.1. A set $P\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ is called attracting (for the trajectory space $\mathcal{H}^+$) if for any bounded in $L_\infty(0,+\infty;E)$ set $B\subset\mathcal{H}^+$ one has

$$\sup_{u\in B}\inf_{v\in P}\|T(h)u-v\|_{C([0,+\infty);E_0)}\underset{h\to\infty}{\to}0.$$

Definition 2.2. A set $P\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ is called absorbing (for the trajectory space $\mathcal{H}^+$) if for any bounded in $L_\infty(0,+\infty;E)$ set $B\subset\mathcal{H}^+$ there is $h\ge 0$ such that for all $t\ge h$:

$$T(t)B\subset P.$$

It is easy to see that any absorbing set is attracting.

Definition 2.3. A set $P\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ is called a trajectory semiattractor (for the trajectory space $\mathcal{H}^+$) if

i) P is compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$;

ii) $T(t)P\subset P$ for any $t\ge 0$;

iii) P is attracting in the sense of Definition 2.1.

Definition 2.4. A set $P\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ is called a trajectory quasiattractor (for the trajectory space $\mathcal{H}^+$) if it satisfies conditions i), iii) of Definition 2.3 and

ii') $T(t)P\supset P$ for any $t\ge 0$.

Definition 2.5. A set $P\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ is called a trajectory attractor (for the trajectory space $\mathcal{H}^+$) if it is a trajectory semiattractor and a trajectory quasiattractor (for the trajectory space $\mathcal{H}^+$). A trajectory attractor is called minimal if it is contained in any other trajectory attractor.

Definition 2.6. A set $\mathcal{A} \subset E$ is called a global attractor (in E_0) for the trajectory space $\mathcal{H}^+$ of equation (2.1) if

i) $\mathcal{A}$ is compact in E_0 and bounded in E;

ii) for any bounded in $L_\infty(0,+\infty;E)$ set $B \subset \mathcal{H}^+$ the attraction property is fulfilled:

$$\sup_{u\in B}\inf_{v\in\mathcal{A}} \|u(t)-v\|_{E_0} \underset{t\to\infty}{\rightarrow} 0$$

iii) $\mathcal{A}$ is the minimal set satisfying conditions i) and ii) (that is $\mathcal{A}$ is contained in every set satisfying conditions i) and ii)).

Remarks. 1. It is obvious that if there exists a minimal trajectory attractor or a global attractor, then it is unique. 2. Definitions 2.1 and 2.6 coincide with the corresponding definitions from [14]. 3. If a trajectory attractor for the trajectory space $\mathcal{H}^+$ is contained in $\mathcal{H}^+$, then it is minimal. It follows from Lemma 2.10 (see below). In [14] (at the additional restriction that $T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ for all $h \geq 0$) there were considered only trajectory attractors (in the sense of Definition 2.5) contained in $\mathcal{H}^+$. Nevertheless, the (more general) concept of minimal trajectory attractor used by us has many usual properties of trajectory attractors. In particular, a minimal trajectory attractor always generates a global attractor (see below, Theorem 2.2). Furthermore, under some conditions on the trajectory space $\mathcal{H}^+$ a minimal trajectory attractor (provided it exists) is always contained in $\mathcal{H}^+$ (see below, Remarks after Theorem 2.5). 4. Definition 2.6 generalizes the well-known concept of (E, E_0)-attractor [2, 5] of the semigroup generated by the Cauchy problem for equation (2.1) under the condition of uniqueness of a solution of this problem (see [14]).

Definition 2.7. The kernel $\mathcal{K}(P)$ of a set $P \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ is the set

$$\{u \in L_\infty(-\infty,+\infty;E) | \forall t \in \mathbb{R} : \Pi_+ T(t)u \in P\}.$$

Here Π_+ is the operator of restriction on the semi-axis $[0,+\infty)$. Obviously, $\Pi_+\mathcal{K}(P) \subset P$.

The concept of kernel allows to estimate the structure of an attractor.

2.2 Simple properties of attracting sets and other auxiliary statements.

Lemma 2.1 *Let a set $P \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ satisfy conditions i), iii) of Definition 2.3. Then $\Pi_+\mathcal{K}(\mathcal{H}^+) \subset P$.*

Roughly speaking, the kernel $\mathcal{K}(\mathcal{H}^+)$ is the set of solutions for equation (2.1) defined on the whole real axis, which are uniformly bounded in E and continuous with values in E_0. The following statement on properties of this set takes place.

Lemma 2.2 *Under the conditions of Lemma 2.1 the kernel $\mathcal{K}(\mathcal{H}^+)$ is relatively compact in $C((-\infty,+\infty);E_0)$ and bounded in $L_\infty(-\infty,+\infty;E)$.*

Trajectory attractors possess the following interesting property.

Lemma 2.3 *If there exists a trajectory attractor P for the trajectory space $\mathcal{H}^+$, then $P = \Pi_+\mathcal{K}(P)$.*

Lemma 2.4 *a) Let sets $P_1, P_2 \subset C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$ satisfy conditions i) or ii) of Definition 2.3. Then $P_1 \cap P_2$ also satisfies a corresponding condition. b) If P_1, P_2 are compact in $C([0, +\infty); E_0)$ and satisfy condition iii) of Definition 2.3, then $P_1 \cap P_2$ also satisfies condition iii).*

Lemma 2.5 *Let a set $P \subset C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$ satisfy one of conditions i), ii), ii') or iii) of Definitions 2.3, 2.4. Then $T(h)P$ also satisfies a corresponding condition for all $h \geq 0$.*

We will need also the following statement.

Lemma 2.6 *Let (X, ρ) be a metric space and $\{K_\alpha\}_{\alpha\in\Xi}$ be a system of non-empty compact sets in X. Assume that for any $\alpha_1, \alpha_2 \in \Xi$ there is $\alpha_3 \in \Xi$ such that $K_{\alpha_1} \cap K_{\alpha_2} = K_{\alpha_3}$. Then $K_0 = \bigcap_{\alpha\in\Xi} K_\alpha \neq \emptyset$ and for any $\epsilon > 0$ there is $\alpha_\epsilon \in \Xi$ such that for any $y \in K_{\alpha_\epsilon}$:*

$$\inf_{x\in K_0} \rho(x, y) < \epsilon.$$

By analogy to the concept of minimal trajectory attractor for the trajectory space $\mathcal{H}^+$ it is possible to introduce the concept of minimal trajectory semiattractor as a trajectory semiattractor contained in any other trajectory semiattractor.

Lemma 2.7 *A minimal trajectory semiattractor is always a minimal trajectory attractor, and vice versa.*

2.3 Existence of a minimal trajectory attractor

Theorem 2.1 *Assume that there exists a trajectory semiattractor P for the trajectory space $\mathcal{H}^+$. Then there exists a minimal trajectory attractor $\mathcal{U}$ for the trajectory space $\mathcal{H}^+$. Here one has $\Pi_+\mathcal{K}(\mathcal{H}^+) \subset \mathcal{U} = \Pi_+\mathcal{K}(\mathcal{U}) \subset \Pi_+\mathcal{K}(P)$ and the kernel $\mathcal{K}(\mathcal{H}^+)$ is relatively compact in $C((-\infty, +\infty); E_0)$ and bounded in $L_\infty(-\infty, +\infty; E)$.*

2.4 Existence of a global attractor

Consider the sections of a trajectory attractor and the kernel at fixed $t \geq 0$:

$$\mathcal{U}(t) = \{v(t) | v \in \mathcal{U}\};$$

$$\mathcal{K}(P)(t) = \{v(t) | v \in \mathcal{K}(P)\}.$$

It is easy to see that these sets are contained in E (see the beginning of Subsection 2.1).

Theorem 2.2 *If there exists a minimal trajectory attractor* $\mathcal{U}$ *for the trajectory space* $\mathcal{H}^+$, *then there is a global attractor* $\mathcal{A}$ *for the trajectory space* $\mathcal{H}^+$ *of equation (2.1) and for all* $t \geq 0$ *one has*

$$\mathcal{K}(\mathcal{H}^+)(t) \subset \mathcal{A} = \mathcal{U}(t) = \mathcal{K}(\mathcal{U})(t).$$

Theorems 2.1 and 2.2 give existence of a global attractor under the conditions of Theorem 2.1. It appears that existence of a global attractor may be proved also under weaker assumptions.

Theorem 2.3 *Assume that there exists a compact in* $C([0, +\infty); E_0)$ *and bounded in* $L_\infty(0, +\infty; E)$ *attracting set* P *for the trajectory space* $\mathcal{H}^+$. *Then there exists a trajectory quasiattractor* $\mathcal{U} \subset P$ *for the trajectory space* $\mathcal{H}^+$ *such that the set* $\mathcal{A} = \mathcal{U}(t)$ *does not depend on* $t \geq 0$ *and is a global attractor for the trajectory space* $\mathcal{H}^+$ *of equation (2.1).*

Remark. For facilitation of check of conditions of Theorems 2.1 and 2.3 the following simple statement may be used.

Lemma 2.8 *Let* P *be a relatively compact in* $C([0, +\infty); E_0)$ *and bounded in* $L_\infty(0, +\infty; E)$ *attracting set for the trajectory space* $\mathcal{H}^+$. *Then its closure* $\overline{P}$ *in* $C([0, +\infty); E_0)$ *is a compact in* $C([0, +\infty); E_0)$ *and bounded in* $L_\infty(0, +\infty; E)$ *attracting set for the trajectory space* $\mathcal{H}^+$. *If in addition* $T(t)P \subset P$ *for any* $t \geq 0$, *then* $\overline{P}$ *is a semiattractor.*

2.5 The case when a trajectory attractor is contained in $\mathcal{H}^+$

Trajectory attractors which are contained in $\mathcal{H}^+$ have additional properties.

Lemma 2.9 *Let* $\mathcal{U} \subset \mathcal{H}^+$ *be a trajectory quasiattractor for the trajectory space* $\mathcal{H}^+$. *Then*

a) $\mathcal{U}$ *is contained in any compact in* $C([0, +\infty); E_0)$ *attracting set* P;

b) in $\mathcal{H}^+$ *there is no trajectory quasiattractors different from* $\mathcal{U}$;

c) if in addition it is known that $T(h)\mathcal{U} \subset \mathcal{H}^+$ *for all* $h \geq 0$, *then* $\mathcal{U}$ *is a minimal trajectory attractor.*

Theorems 2.1–2.3 and Lemma 2.10 imply

Theorem 2.4 *Assume that there exists a compact in* $C([0, +\infty); E_0)$ *and bounded in* $L_\infty(0, +\infty; E)$ *attracting set* P *for the trajectory space* $\mathcal{H}^+$. *Let* $T(h)P \subset \mathcal{H}^+$ *for all* $h \geq 0$. *Then there exist a minimal trajectory attractor* $\mathcal{U} = \Pi_+\mathcal{K}(\mathcal{H}^+)$ *for the trajectory space* $\mathcal{H}^+$ *and a global attractor* $\mathcal{A} = \mathcal{U}(t) = \mathcal{K}(\mathcal{H}^+)(t)$, $t \geq 0$ *for the trajectory space* $\mathcal{H}^+$ *of equation (2.1).*

Remarks. 1. Under conditions of Theorem 2.4 $\mathcal{U} = \Pi_+\mathcal{K}(\mathcal{H}^+)$, so the kernel $\mathcal{K}(\mathcal{H}^+)$ is non-empty. 2. In [14] (Theorem 1.1) this theorem was proved under additional assumption $T(h)\mathcal{H}^+ \subset \mathcal{H}^+$ for all $h \geq 0$.

2.6 Structure of the minimal trajectory attractor

Here we give a theorem which gives some characterization for the structure of a minimal trajectory attractor and helps to specify the connection of this concept with the trajectory attractor from [14].

Let us define a topology on the set $C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ as follows: a set $V\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ is closed if the limit of any bounded in $L_\infty(0,+\infty;E)$ and converging in $C([0,+\infty);E_0)$ sequence of elements from V belongs to V. The closure of a set in this topology is denoted by square brackets $[\cdot]$.

Lemma 2.10 *For every fixed $h\geq 0$, the map*

$$T(h): C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)\to C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$$

is continuous. As a corollary, for any $V\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ one has

$$T(h)[V]\subset[T(h)V].$$

Theorem 2.5 *Assume that there exists a minimal trajectory attractor $\mathcal{U}$ for the trajectory space $\mathcal{H}^+$. Then*

$$\Pi_+\mathcal{K}(\mathcal{H}^+)\subset\mathcal{U}\subset\Pi_+\mathcal{K}([\bigcup_{t\geq 0}T(t)\mathcal{H}^+]) \tag{2.6}$$

Remarks. 1. Let $T(h)\mathcal{H}^+\subset\mathcal{H}^+$ at all $h\geq 0$ and $[\mathcal{H}^+]=\mathcal{H}^+$. If there exists a minimal trajectory attractor $\mathcal{U}$ for the trajectory space $\mathcal{H}^+$, then by Theorem 2.5 $\mathcal{U}=\Pi_+\mathcal{K}(\mathcal{H}^+)\subset\mathcal{H}^+$, that is $\mathcal{U}$ is a trajectory attractor in the sense of [14]. Using Theorem 2.2 or Theorem 2.4 we conclude that $\mathcal{A}=\mathcal{U}(t)=\mathcal{K}(\mathcal{H}^+)(t)\quad(t\geq 0)$ is a global attractor for the trajectory space $\mathcal{H}^+$ of equation (2.1). 2. In [14] there was investigated existence of attractors for the Navier-Stokes problem. The set of weak solutions of the Navier-Stokes problem satisfying an energy estimate of differential type was taken as a trajectory space $\mathcal{H}^+$. In this situation it appears that $T(h)\mathcal{H}^+\subset\mathcal{H}^+$ for all $h\geq 0$ and $[\mathcal{H}^+]=\mathcal{H}^+$ ([14], statement 3.3).

3 Weak solutions of boundary value problem for the Jeffreys model

In this section we shall describe the weak setting of problem (0.1) - (0.4) and give an existence theorem for solutions of this problem.

Definition 3.1. Let $f\in V^*$. A weak solution of problem (0.1) - (0.4) on an interval $(0,T)$, $T>0$ is a pair of functions (u,σ),

$$u\in L_2(0,T;V)\bigcap C_w([0,T];H),\ \frac{du}{dt}\in L_1(0,T;V^*),$$

$$\sigma \in L_2(0,T;L_2(\Omega,\mathbb{R}_S^{N\times N}))\bigcap C_w([0,T];H^{-1}(\Omega,\mathbb{R}_S^{N\times N})) \tag{3.1}$$

satisfying identities

$$\frac{d}{dt}(u,\varphi)+(\sigma,\nabla\varphi)-\sum_{i=1}^{n}(u_iu,\frac{\partial\varphi}{\partial x_i})=\langle f,\varphi\rangle \tag{3.2}$$

$$(\sigma,\Phi)+\lambda_1\frac{d}{dt}(\sigma,\Phi)-\lambda_1\sum_{i=1}^{n}(u_i\sigma,\ \frac{\partial\Phi}{\partial x_i})=$$

$$-2\eta(u,Div\Phi)-2\eta\lambda_2(\frac{d}{dt}(u,Div\Phi)+\sum_{i=1}^{n}(u_i\mathcal{E}(u),\frac{\partial\Phi}{\partial x_i})) \tag{3.3}$$

for all $\varphi\in\mathcal{V}$ and $\Phi\in C_0^\infty$ in the sense of distributions on $(0,T)$.

Let us introduce the notations

$$\mu_1=\eta\frac{\lambda_2}{\lambda_1},\ \mu_2=\frac{\eta-\mu_1}{\lambda_1},\ \tau=\sigma-2\mu_1\mathcal{E} \tag{3.4}$$

With the help of these notations we can rewrite (3.3) and (3.2) in the following form:

$$\frac{d}{dt}(\tau,\Phi)+\frac{1}{\lambda_1}(\tau,\Phi)-\sum_{i=1}^{n}(u_i\tau,\frac{\partial\Phi}{\partial x_i})+2\mu_2(u,Div\Phi)=0 \tag{3.5}$$

$$\frac{d}{dt}(u,\varphi)-\sum_{i=1}^{n}(u_iu,\frac{\partial\varphi}{\partial x_i})+\mu_1(\nabla u,\nabla\varphi)+(\tau,\nabla\varphi)=\langle f,\varphi\rangle \tag{3.6}$$

for all $\varphi\in\mathcal{V}$ and $\Phi\in C_0^\infty$.

For research of attractors for weak solutions of problem (0.1) - (0.4) it appears to be convenient to pass to the variables (u,τ) and to investigate attractors of problem (3.5) - (3.6).

Now we state a theorem of existence of solutions for these problems.

Theorem 3.1 *Let $f\in V^*$. Given $a\in H$, $\tau_0\in L_2(\Omega,\mathbb{R}_S^{n\times n})$, there is a pair of functions (u,τ) which*

i) belongs to the class

$$u\in L_2(0,T;V)\bigcap L_\infty(0,T;H)\bigcap C_w([0,T];H),$$

$$u'\in L_{4/3}(0,T;V^*),$$

$$\tau\in L_\infty(0,T;L_2)\bigcap C_w([0,T];L_2),\tau'\in L_2(0,T;H^{-2}); \tag{3.7}$$

ii) satisfies the initial condition

$$u|_{t=0}=a,\ \ \tau|_{t=0}=\tau_0; \tag{3.8}$$

iii) satisfies identities (3.5), (3.6) a.e. on $(0,T)$ *for all* $\varphi \in \mathcal{V}$ *and* $\Phi \in C_0^\infty$*;*
iv) satisfies the energy inequality:

$$\frac{1}{2}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{4\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;V)} \leq$$

$$\leq e^{-2\gamma t}(\|a\|^2 + \frac{1}{2\mu_2}\|\tau_0\|^2) + \frac{\gamma+1}{2\mu_1\gamma}\|f\|^2_{V^*} \tag{3.9}$$

for $t \in [0, T-1]$*. Here* $\gamma = \min(\frac{1}{\lambda_1}, \frac{\mu_1}{2K_0(\Omega)^2})$*, where* K_0 *is the constant from inequality (1.1).*

Remarks. 1. It is clear that under the conditions of Theorem 3.1 the pair (u, σ), $\sigma = \tau + 2\mu_1\mathcal{E}(u)$ is a weak solution of problem (0.1) - (0.4) in the sense of Definition 3.1. 2. We consider V to be equipped with the norm $\|\cdot\|_1$ introduced above and V^* to be equipped with the corresponding norm of a conjugate space.

4 Attractors for weak solutions of boundary value problem for the Jeffreys model

In this section we construct the minimal trajectory attractor and the global attractor for problem (3.5) - (3.6). We choose $H \times L_2(\Omega, \mathbb{R}^{n\times n}_S)$ as the space E and the space $V^*_\delta \times H^{-\delta}(\Omega, \mathbb{R}^{n\times n}_S)$ as the space E_0, where $\delta \in (0,1]$ is a fixed number. As the trajectory space $\mathcal{H}^+$ for the Jeffreys model we take the set of pairs of functions (u,τ) which

i) belong to the class

$$u \in L_{2,loc}(0,+\infty;V)\bigcap L_\infty(0,+\infty;H)\bigcap C_w([0,+\infty);H),$$

$$\tau \in L_\infty(0,+\infty;L_2)\bigcap C_w([0,+\infty);L_2); \tag{4.1}$$

ii) satisfy identities (3.5), (3.6) a.e. on $(0,+\infty)$ for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$;
iii) satisfy the energy inequality:

$$\frac{1}{2}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{4\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;V)} \leq$$

$$\leq e^{-2\gamma t}(\|u\|^2_{L_\infty(0,+\infty;H)} + \frac{1}{2\mu_2}\|\tau\|^2_{L_\infty(0,+\infty;L_2)}) + \frac{\gamma+1}{2\mu_1\gamma}\|f\|^2_{V^*} \tag{4.2}$$

for all $t \geq 0$ where γ is as in Theorem 3.1.

Remarks. 1. In section 2 it was supposed that $\mathcal{H}^+ \subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$. Let us show that this condition holds for the Jeffreys model. In fact, every pair (u,τ) from $\mathcal{H}^+$ belongs to $L_\infty(0,+\infty;H\times L_2) = L_\infty(0,+\infty;E)$. For any $T \geq 0$ the function $u \in L_2(0,T;V)$. By Lemma 3.2 $u' \in L_{4/3}(0,T;V^*)$, $\tau' \in L_2(0,T;H^{-2})$. Since Ω is bounded, $H_0^\delta \subset L_2$ compactly (see e.g. [12], Theorem 1.1 of

chapter II). Therefore $V_\delta \subset H$ compactly, $L_2 \subset H^{-\delta}$ compactly, $H \subset V_\delta^*$ compactly. By the Aubin-Simon compactness theorem (see [10], Corollary 4) one has:
$u \in C([0,T]; V_\delta^*)$, $\tau \in C([0,T]; H^{-\delta})$, that is $(u, \tau) \in C([0,T]; E_0)$ for any $T \geq 0$.
2. It is clear that on account of inequality (4.2) the trajectory space $\mathcal{H}^+$ for the Jeffreys model is not invariant with respect to the translation operator $T(h)$.

Theorem 4.1 *Let $f \in V^*$. Given $a \in H$, $\tau_0 \in L_2(\Omega, \mathbb{R}_S^{n\times n})$, there is a pair of functions (a trajectory) $(u, \tau) \in \mathcal{H}^+$ satisfying initial condition (3.8) .*

The main result of this section is

Theorem 4.2 *Let $f \in V^*$. There exists a minimal trajectory attractor $\mathcal{U}_J$ for the trajectory space $\mathcal{H}^+$ and (2.6) is fulfilled.*

Theorems 2.2 and 4.2 imply

Theorem 4.3 *Let $f \in V^*$. In the space $H \times L_2$ there is a global attractor $\mathcal{A}_J$ for problem (3.5)-(3.6), i.e. a minimal compact in $V_\delta^* \times H^{-\delta}$ and bounded in $H \times L_2$ set, which attracts all trajectories from $\mathcal{H}^+$ (see Definition 2.6). For all $t \geq 0$:*

$$\mathcal{K}(\mathcal{H}^+)(t) \subset \mathcal{A}_J = \mathcal{U}_J(t) = \mathcal{K}(\mathcal{U}_J)(t).$$

Remarks. 1. We have established existence of minimal trajectory and global attractors for the space $\mathcal{H}^+$ of weak solutions for problem (0.1) - (0.4) on the positive axis, which satisfy the energy inequality (4.2). At the same time it is not known whether there exist weak solutions of this problem which do not satisfy the energy inequality. Such problem is open even for the Navier-Stokes system. 2. Similar results were obtained recently by us for non-autonomous equations.

References

[1] Y.Y. Agranovich, V.G. Zvyagin. Investigations of Properties of Attractors for a Regularized Model of the Motion of a Nonlinear-Viscous Fluid, International Journal of Mathematics, Game Theory and Algebra, 2001, 12(1), 71-87.

[2] Babin A.V., Vishik M.I. Attractors of evolution equations. Nauka, Moscow, 1989.

[3] V.V. Chepyzhov, M.I. Vishik, Evolution equations and their trajectory attractors, J. Math Pures Appl. 76 (1997), 913-964.

[4] Gyarmati I. Non-equilibrium thermodynamics. Field theory and variational principles, Heidelberg, Springer, 1970.

[5] Hale J., Asymptotic behavior of dissipative systems, Mathematical surveys and Monographs, 25, American Math. Soc., Providence, R.I., 1988.

[6] Kotsiolis A.A., Oskolkov A.P. On the limiting conditions and an attractor for the motion equations of Oldroyd's fluid, Zap. nauch. sem. LOMI, 1986, V. 152, 67-71.

[7] Ladyzenskaya O.A. On the dynamic system generated by Navier-Stokes equations, Zap. nauch. sem. LOMI, 1972, V. 27, 91-115.

[8] Reiner M. Reology. In: Handbuch der Physik, S. Flugge, ed., Bd. VI, Springer, 1958.

[9] G. Sell, Global attractors for the three-dimensional Navier-Stokes equations, J. Dyn. Diff. Eq., 1996, V. 8, No. 1, 1-33.

[10] Simon J. Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl.-1987.-V. 146.-P. 65-96.

[11] Simon J. On the existence of the pressure for solutions of the variational Navier-Stokes equation, J. Mathematical Fluid Mech., 3 (1999), 225-234.

[12] R.Temam. Navier-Stokes equations. Theory and numerical analysis, North-Holland, 1979.

[13] R.Temam. Infinite-dimensional dynamical systems in mechanics and physics. 2nd edition. Springer-Verlag, 1997.

[14] Vishik M.I., Chepyzhov V.V. Trajectory and global attractors for 3D Navier-Stokes system, Mat. Zametki, 2002, V. 71, 194-213.

[15] D.A. Vorotnikov, V.G. Zvyagin, On the existence of weak solutions for the initial-boundary value problem in the Jeffreys model of motion of a viscoelastic medium, Abstr. Appl. Anal., 2004. V. 2004, # 10, 815-829.

[16] Zvyagin V.G., Dmitrienko V.T. Approximating-topological approach to investigation of problems of hydrodynamics. Navier-Stokes system, Editorial URSS, Moscow, 2004.