Research of mathematical model of low concentrated aqueous polymer solutions.

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1 Introduction

We study the following initial-boundary value problem in a bounded domain $\Omega \subset \mathbf{R}^n, n = 2, 3$ with locally-Lipschitz boundary on a time interval $[0, T], (T < \infty)$:

$$\frac{\partial v}{\partial t} - \nu \Delta v + v_i \frac{\partial v}{\partial x_i} - \varkappa \frac{\partial \Delta v}{\partial t} - \varkappa \operatorname{Div} \left(v_k \frac{\partial \mathcal{E}(v)}{\partial x_k} \right) + \operatorname{grad} p = f, \quad x \in \Omega,\ t \in [0, T]; \tag{1.1}$$

$$\operatorname{div} v = 0, \quad x \in \Omega,\ t \in [0, T], \tag{1.2}$$

$$v(x, 0) = a(x), \quad x \in \Omega, \tag{1.3}$$

$$v|_{\partial\Omega \times [0,T]} = 0. \tag{1.4}$$

Hereinafter Einstein's summation convention is supposed. The given system of equations differs from the Navier-Stokes system of equations by presence of the additional term $\varkappa \operatorname{Div} \frac{d\mathcal{E}}{dt}$. This term describes relaxational properties of a fluid. In the case of very weak relaxational properties of the fluid (for $\varkappa$ close to zero), and also in the case when the motion of the liquid has steady-state character (the total derivative of the strain rate tensor with respect to time is equal to zero) this additional term vanishes. In this case the system of equations (1.1),(1.2) coincides with the system of Navier-Stokes equations. But in the case of turbulent behavior of the fluid and on unsteady laminar behavior of the fluid this additional term is nonzero and must play a significant role. Just so low concentrated aqueous solutions of polymers behave and it has been confirmed by experimental researches of polyethylenoxide and polyacrylamide aqueous solutions and guar gum aqueous solutions. Therefore the given model has also received the title of the mathematical model of motion of low concentrated aqueous polymer solutions.

2 Principal notations and functional spaces.

Let us introduce the following functional spaces used hereinafter:

$\mathfrak{D}(\Omega)^n$ is the space of smooth functions on Ω with values in $\mathbb{R}^n$ and with compact support contained in Ω;

$\mathcal{V} = \{v : v \in \mathfrak{D}(\Omega)^n, \operatorname{div} v = 0\}$ is the set of solenoidal smooth functions;

V is the completion of $\mathcal{V}$ in the norm of $W_2^1(\Omega)^n$; we consider the space V with the following norm

$$\|v\|_V = \left(\int_\Omega \nabla v : \nabla v \, dx \right)^{\frac{1}{2}}.$$

Here $\nabla u : \nabla \varphi$, $u = (u_1, ..., u_n)$, $\varphi = (\varphi_1, ..., \varphi_n)$ denotes the component-wise multiplication of matrixes:

$$\nabla u : \nabla \varphi = \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial \varphi_i}{\partial x_j}.$$

In V this norm is equivalent to the norm induced from the space $W_2^1(\Omega)^n$.

X is the completion of $\mathcal{V}$ in the norm of $W_2^3(\Omega)^n$; the norm in X is defined by

$$\|v\|_X = \left(\int_\Omega \nabla(\Delta v) : \nabla(\Delta v) \, dx \right)^{\frac{1}{2}}.$$

In X this norm is equivalent to the norm induced from the space $W_2^3(\Omega)^n$.

By X^* we denote the space conjugate to the space X. Denote by $\langle h, v\rangle$ the value of a functional $h \in X^*$ on a function $v \in X$.

Denote by Z the image of X under the action of the operator $(I - \varkappa\Delta)$, i.e. $Z = (I - \varkappa\Delta)X$. Let us note that Z is a subspace of V but does not coincide with it.

Now we can introduce principal functional space used below.

$$E_1 = \{v : \, v \in L_\infty(0, T; V), \; v' \in L_2(0, T; Z^*)\}$$

with the norm:

$$\|v\|_{E_1} = \|v\|_{L_\infty(0,T;V)} + \|v'\|_{L_2(0,T;Z^*)}.$$

2.1 Definition of weak solution for initial-boundary value problem (1.1)-(1.4).

Suppose that $f \in L_2(0, T; V^*), a_* \in V$.

Definition 2.1. *A function $v \in E_1$ is a weak solution for the initial-boundary value problem* (1.1)-(1.4) *if for any $\varphi \in X$ and almost all $t \in (0, T)$ the function v satisfies the equality*

$$\Big\langle (I - \varkappa\Delta)\frac{\partial v}{\partial t}, \varphi \Big\rangle - \int\limits_\Omega v_i v_j \frac{\partial \varphi_j}{\partial x_i}\, dx + \nu \int\limits_\Omega \nabla v : \nabla \varphi \, dx -$$

$$- \frac{\varkappa}{2} \int\limits_\Omega v_k \frac{\partial v_i}{\partial x_j} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k}\, dx - \frac{\varkappa}{2} \int\limits_\Omega v_k \frac{\partial v_j}{\partial x_i} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k}\, dx = \langle f, \varphi\rangle \quad (2.1)$$

and the initial condition

$$v(0) = a_*. \quad (2.2)$$

2.2 The main result.

Our main result is the following theorem:

Theorem 2.1. *For any $f \in L_2(0, T; V^*), a \in V$ the initial-boundary value problem* (1.1)-(1.4) *has at least one weak solution $v_* \in E_1$.*

In order to prove this theorem we use the modified method of introduction of auxiliary viscosity. At the first stage we introduce into the equation (1.1) the term $-\varepsilon\Delta^3\left(\frac{\partial v}{\partial t}\right)$. The resolvability of the obtained approximating problem is established by topological methods on the basis of a priori estimates of weak solutions. Then it is shown that in a sequence of weak solutions of the approximating problem it is possible to select a subsequence converging to a weak solution of the initial-boundary value problem (1.1)-(1.4) as the parameter of approximation tends to zero.