An exact bounded PML technique for the scattering of acoustic waves

A. Bermúdez [1] L. Hervella-Nieto[2] A. Prieto[1] **R. Rodríguez**[3]

1 Departamento de Matemática Aplicada. Universidad de Santiago de Compostela. España.
2 Departamento de Matemáticas. Universidad de A Coruña. España
3 GI^{2}MA, Departamento de Ingeniería Matemática. Universidad de Concepción. Chile.

Abstract

We consider the following Helmholtz problem which models the propagation of a wave of frequency $\omega > 0$ and velocity of propagation $c > 0$ in an unbounded homogeneous medium:

$$\begin{cases} \Delta u + k^2 u = 0 \qquad \text{in } \Omega_{\mathrm{E}}, \\ u = g \qquad \text{on } \Gamma, \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u}{\partial r} - iku\right) = 0, \end{cases}$$

where $k := \omega/c$ is the wave number, $\Omega_{\mathrm{E}} := \mathbb{R}^2 \setminus \overline{\Omega}_{\mathrm{I}}$, with $\Omega_{\mathrm{I}} \subset \mathbb{R}^2$ being a simply connected bounded domain with regular boundary Γ, and $g \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$ is a given source function. The third equation is a typical *Sommerfeld* condition modeling the radiation of the wave at infinity. This is is a classical scattering problem, whose existence and uniqueness of solution is well known (see for instance [4]).

The typical first step for the numerical solution by finite elements or finite differences of such a scattering problem is to truncate the unbounded computational domain, which entails an inherent difficulty: *how to choose boundary conditions to replace the Sommerfeld radiation condition at infinity* (see for instance [5]). There are several techniques to deal with this: boundary element methods, infinite element methods, Dirichlet-to-Neumann methods based on Fourier expansions, or the use of absorbing boundary conditions.

An alternative approach to deal with the truncation of unbounded domains is the so called *PML* (Perfectly Matched Layer) technique, introduced by Berenger [1] for Maxwell's equations in electromagnetism. It is based on simulating an absorbing layer of damping material surrounding the domain of interest, like a thin sponge which absorbs the scattered field radiated to the exterior of the domain. The absorbing material is characterized by a damping function varying through the thickness of the layer. Typical examples are linear and quadratic damping functions ([1, 3]).

In a recent paper [2], we have introduced an 'exact' bounded PML, based on using a singular damping function. 'Exactness' must be understood in the sense that this technique allows exact recovering of the solution to time-harmonic scattering problems in unbounded domains. In spite of the singularity of the damping function, the procedure is shown to lead to a well posed conforming finite element discretization.

First, we analyze this approach in a simplified one-dimensional framework, which allows us to choose a convenient singular damping function. Subsequently, it is proved that a similar damping function acting on an annular layer also leads to an exact recovery of the

solution on the physical domain. Finally, we report some numerical tests exhibiting the high accuracy of this technique, as well as its advantages as compared with other classical PML methods.

References

[1] J.P. Berenger, A perfectly matched layer for the absorption of electromagnetic waves, *J. Comput. Phys.*, **114** (1994) 185-200.

[2] A. Bermúdez, L. Hervella-Nieto, A. Prieto, R. Rodríguez. An exact bounded PML for the Helmholtz equation. *C. R. Acad. Sci. Paris, Serie I*, **339** (2004) 803-808.

[3] F. Collino, P. Monk, The perfectly matched layer in curvilinear coordinates. *SIAM J. Sci. Comput.*, **19** (1998) 2061-2090.

[4] D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory*, John Wiley, New York, 1983.

[5] D. Givoli, *Numerical Methods for Problems in Infinite Domains*, Elsevier, Amsterdam, 1992.