ON THE DETERMINATION THE POTENTIAL-ENERGY FUNCTION FROM GIVEN ORBITS

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1. Introduction

One of the fundamental classical problems in celestial mechanics is to determine the potential-energy function U such that every curve from a given family of curves will be a possible trajectory of a particle moving under the action of potential forces F, admitting U; i. e. $F = gradU$.

The importance of this problem was already acknowledged by Bozis [Boz] and Szebehely [Sze1]. Szebehely, indeed, affirms that in order to establish accurate physical descriptions and accurate constants, one needs to address the inverse problem of dynamics.

The first inverse problem in Celestial Mechanics was stated and solved by Newton (1687) [New] and concerns the determination of the potential field of force that ensures the planetary motion in accordance to the observed properties, namely to Kepler's laws.

Bertrand (1877) [Ber] proved that the expression for Newton's force of attraction can be obtained directly from the Kepler first law to within a constant multiplier.

Bertrand stated also a more general problem of determining a positional force, under which a particle describes a conic section under any initial conditions.

The ideas of Bertrand were developed by Dainelli [Dai], Suslov [Sus], Joukovski [Jou],
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Dainelli in [Dai] essentially states a more general problem of how to determine the most general field of force (the force being supposed to depend only on the position of the particle on which it acts) under which a given family of planar curves is a family of orbits of a particle.

The solution proposed by Dainelli is the following [Whi,Dain,Sad].

The most general field of force $\mathbf{F} = (F_x,\, F_y)$ capable of generating the family of planar orbits $f(x, y) = Const$ is the following:

$$\begin{cases} F_x = -\lambda^2\{f, \partial_y f\} - \lambda\{f,\, \lambda\}\partial_y f \\ F_y = \lambda^2\{f, \partial_x f\} + \lambda\{f,\, \lambda\}\partial_x f. \end{cases} \tag{1.1}$$

where λ is an arbitrary function.

In [Sus], Suslov stated and solved a problem which was a further development of Bertrand's problem. He shows that, given a $(N-1)$-parametric family of orbits in the configuration space of a holonomic system with N degrees of freedom and a kinetic energy T, it is necessary to determine the potential field of force under which any trajectory of the family can be traced by the representative point of the system.

Suslov deduced the following system of linear partial differential equations with respect to the require potential function:

$$\frac{\partial\theta}{\partial\triangle_k}\frac{\partial U}{\partial x^N}-\frac{\partial\theta}{\partial\triangle_N}\frac{\partial U}{\partial x^k}=\frac{U+h}{\theta}\Big(\frac{\partial\theta}{\partial\triangle_N}\frac{\partial\theta}{\partial x^k}-\frac{\partial\theta}{\partial\triangle_k}\frac{\partial\theta}{\partial x^N}+\sum_{m=1}^{N}\triangle^m(\frac{\partial\theta}{\partial\triangle_k}\frac{\partial^2\theta}{\partial\triangle_N\partial x^m}-\frac{\partial\theta}{\partial\triangle_N}\frac{\partial^2\theta}{\partial\triangle_k\partial x^m})\Big)$$

where θ, $\triangle^1$, $\triangle^2, ..., \triangle^N$ are functions:

$$\sum_{j=1}^{N}\frac{\partial f_\alpha}{\partial x^j}\triangle^j=0,\quad \triangle_k=\sum_{j=1}^{N}G_{jk}(x)\triangle^j,\quad \alpha=1,2,..,N-1,\ k=1,2,..,N.$$

$$\theta=\frac{1}{2}\sum_{k,j=1}^{N}G_{kj}(x)\triangle^k\triangle^j\equiv\theta(x^1,x^2,..,x^N,\triangle_1,\ \triangle_2,...,\triangle_N)$$

and proved that theses equations represented the necessary and sufficient conditions under which the equations of motion of the study mechanical system admits the given $N-1$ partial integrals.

Assuming that given trajectories admit a family of the orthogonal surfaces, Joukovski in [Jou] constructed the potential-energy functions in explicit forms for systems with two and three degrees of freedom.

The following theorem was enunciated by Joukovsky in 1890, that

if $q = const$ is the equation of the family of curves on a surface, and $p = const$ denotes the family of curves orthogonal to these, then the curves $q = const$ can be freely described by a particle under the influence of forces derived from the potential-energy function

$$V=\Delta_1(p)\Big(g(p)+\int h(q)\frac{\partial}{\partial q}(\frac{1}{\Delta_1(p)})dq\Big)$$

where h and g are arbitrary functions, and Δ_1 denotes the first differential parameter

In the most general form the inverse problem in dynamics was studied in [Sad]. By applying the results presented in that work we propose the following new results:

1. Generalization the Dainelli problem for a mechanical system with N degree of freedom
2. New approach to solve the Suslov problem
3. Generalization of the Joukovski problem for a mechanical system with N degree of freedom

2. Solution of the generalized Dainelli problem

Definition 2.1 [Generalized Dainelli's problem].

Given a $N-1$ -parametric family of orbits in the configuration space of a holonomic system with N degrees of freedom and kinetic energy $T=\frac{1}{2}\sum_{j,k=1}^{N}G_{kj}(x)\dot{x}^j\dot{x}^k$. The Generalized Dainelli problem is the problem of determining the most general field of force that depends only on the position of the system under which any trajectory of the family can be traced by a representative point of the system.

Proposition 2.1.

Given a mechanical system $\mathcal{M}$ with configuration space X and a kinetic energy T. Then the most general field of force that depends only on the position of the system and is capable of generating the given orbits $f_j(x) = c_j, \quad j = 1, \dots, N-1$ is described by the equation

$$\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^n} - \frac{\partial T}{\partial\, x^n} = \omega(\partial_n), \quad n = 1, 2, .., N \\ &\omega = d\frac{\|\mathbf{v}\|^2}{2} + \lambda \sum_{j=1}^{N-1} a_{Nj} df_j. \end{aligned} \tag{2.1}$$

Here $\imath_{\mathbf{v}}$ is the contraction along the vector field $\mathbf{v}$, $f_1, \dots, f_{N-1}$ are independent functions of class $C^r(\widetilde{X} \subseteq X)$, $r \geq 2$, $\mathbf{v}$ is the vector field

$$\mathbf{v} = \lambda \begin{vmatrix} df_1(\partial_1) & \dots & df_1(\partial_N) \\ \vdots & & \vdots \\ df_{N-1}(\partial_1) & \dots & df_{N-1}(\partial_N) \\ \partial_1 & \dots & \partial_N \end{vmatrix} \equiv \lambda\{f_1, \dots, f_{N-1}, *\}, \tag{2.2}$$

λ is an arbitrary function, $\partial_j \equiv \dfrac{\partial}{\partial x_j}$, and σ is a 1-form associated with $\mathbf{v}$, so that

$$\begin{cases} a_{Nj} = (-1)^{N+j-1} d\sigma \wedge df_1 \wedge df_2 \wedge \dots \wedge df_{j-1} \wedge df_{j+1} \wedge \dots \wedge df_{N-1}(\partial_1, \dots, \partial_N) \\ a_{N1} = (-1)^N d\sigma \wedge df_2 \wedge \dots \wedge df_{N-1}(\partial_1, \dots, \partial_N) \end{cases} \tag{2.3}$$

where $j = 2, \dots, N-1$.

Corollary 2.1. The field of force (2.1) takes for a particle in R^2 the following forms:

$$\omega = d\frac{1}{2}\lambda^2((\partial_x f)^2 + (\partial_y f)^2) + \lambda\big(\partial_x(\lambda f_x) + \partial_y(\lambda f_y)\big) df \tag{2.4}$$

It is possible to show that (2.4) coincide with (1.1) [Sad].

In the next section we make use of the solution of the generalized Dainelli inverse problem for studying particular cases of the Suslov and of the generalized Joukovski problems.

Proposition 2.2. The field of force (2.1) is potential, i.e., $\omega = -dU(x)$ iff

$$\lambda \sum_{j=1}^{N-1} a_{Nj}(x) df_j = -dh(f_1, f_2, \dots, f_{N-1}) \tag{2.5}$$

Corollary 2.2 The field of force (2.1) for a particle in R^2 is potential iff

$$\lambda\big(\partial_x(\lambda f_x) + \partial_y(\lambda f_y)\big) df = dh(f)$$

and respectively in R^3, iff

$$\lambda \imath_{\,curl\,\mathbf{v}}(df_1 \wedge f_2) = dh(f_1, f_2)$$

where $\mathbf{v} = \lambda(x,y,z)(\,grad\, f_1 \times \,grad\, f_2)$

In 1974 Szebehely [Sze2] obtained a linear first-order partial differential equation for the potential function U which gives rise to a one-parameter family of planar orbits with a given total energy h. This result originated many works on the inverse problems (see for instance [Boz]). The equation of Szebehely was generalized to a two-parameter family of three-dimensional orbits by Bozis (1983) .

We show that the results, presented in those works, can be obtained from the solutions of the Suslov problem.

By aplying the above results we prove the following result

Corollary 2.3 (Solution of the Bertrand Problem)

The potential-energy function U capable of generating a one-parameter family of conics with eccentricity b is the function

$$U = a_{-1}(H_1(\cos\theta - K_1 \log r(1+b\cos\theta)) + \sum_{j\in \mathrm{Z}\setminus\{-1\}} a_j r^{j+1}(H_j(\cos\theta) - \frac{1+b\cos\theta}{j+1})$$

where $a_j \quad j \in \mathrm{Z}$, K_1 are real constants and H_j, $j \in \mathrm{Z}$ are solutions of the Heun equations with singularities at the points

$$0,\, 1\,, \frac{1+b}{b}, \infty$$

and with the exponents

$$(0, \frac{j+3+b(j+1)}{2b});\ (0, j - \frac{j+3+b(j+1)}{2b});\ (0, j+1); (-1-j, 1-j)$$

respectively.

Definition [Generalised Jukovski problem]. The generalized Joulovski problem is a particular case of the Suslov problem, which is obtained by assuming that the vector field (2.2) has the form :

$$\{f_1, f_2, \ldots, f_{N-1}, * \quad \} = \nu(x)\nabla f_N, \tag{2.6}$$

where $\nabla f_N = \sum_{j=1}^{N} G^{jk}(x)\partial_j f_N \partial_k, \quad G^{-1} = (G^{jk})$ is the inverse matrix of the Riemann metric G and ν is a certain function.

The stated problem coincides with Joukovski's problem when $N = 3$. [Whit, Jou]

Proposition 2.3 The field of force expressed by equations (2.1) and (2.6) is potential iff

$$\lambda\nu i_{\,grad\, f_N}\big(d(\lambda\nu df_N)\big) = -dh(f_1, f_2, \ldots, f_{N-1})$$

Clearly, if $\nu\lambda_N = \Gamma(f_N)$ then the required potential-energy function U is

$$U = \frac{1}{2}\Gamma(f_N)\nabla f_N(f_N) - h_0$$

We illustrate this result by determining a solution of the inverse problem which we will call the inverse Stäckel problem.

We prove the following proposition which represented an extension of the Joukovski theorem for a mechanical system with N degree of freedom.

Proposition 2.4

If

$$x^j = C_j = const, \quad j = 1, 2, .., N-1$$

are the equations of the $N-1$ parametric family of curves on X, and $x^N = const$ denotes the family of curves orthogonal to these, then the curves $x^j = C_j = const$ can be freely described by a representative particle under the influence of forces derived from the potential-energy function

$$U = G_{NN}(x^1, x^2, .., x^N)\Big(g(x^N) + \sum_{j=1}^{N-1} \int h(x^1, x^2, .., x^{N-1}) \frac{\partial G_{NN}(x^1, x^2, ..x^N)}{\partial x^j} dx^j\Big)$$

where h and g are arbitrary functions.

Clearly, for $N = 2$ we exactly obtain the Joukovski result given in the introduction.

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