Approximation and Complexity Trade-off in Controller Design: a Case Study of the TORA system

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***Abstract*— The main objective of the paper is to study the approximation and complexity trade-off capabilities of the recently proposed Tensor Product Distributed Compensation (TPDC) based control design framework. The TPDC is the combination of the TP model transformation and the Parallel Distributed Compensation (PDC) framework. The TP model transformation includes an HOSVD based technique to solve the approximation and complexity trade-off. In this paper we generate TP models with different complexity and approximation properties, and then we derive controllers for them. We analyze how the trade-off effects the model behavior and control performance. All these properties are studied via the state feedback controller design of the Translational Oscillations with an Eccentric Rotational Proof Mass Actuator (TORA) System.**

***Index Terms*— Nonlinear control, Tensor Product Distributed Compensation framework (TPDC), Linear Matrix Inequalities (LMI), TORA system.**

I. Introduction

THE TP model form is a dynamic model representation whereupon Linear Matrix Inequality (LMI) based control design techniques [1]–[3] can immediately be executed. It describes a class of Linear Parameter Varying (LPV) models in a polytopic form that is the convex combination of linear time invariant (LTI) models, where the convex combination is defined by the weighting functions of each parameter separately. This model is called TP model, or polytopic model.

The TP model transformation is a recently proposed numerical method to transform LPV models into TP model form (polytopic form) [4], [5]. It is capable of transforming different LPV model representations (such as physical model given by analytic equations, fuzzy, neural network, genetic algorithm based models) into TP model form in a uniform way. In this sense it replaces the analytical derivations and affine decompositions (that could be a very complex or even an unsolvable task), and automatically results in the TP model form. Execution of the TP model transformation takes a few minutes by a regular Personal Computer. The TP model transformation minimizes the number of the LTI components of the resulting TP model. Furthermore, the TP model transformation is capable of resulting different convex hulls of the given LPV model.

One can find a number of LMIs under the PDC framework which can immediately be applied to the TP model, according to various control design specifications. Therefore, it is worth linking the TP model transformation and the PDC design framework [6]. That is called Tensor Product Distributed Compensation (TPDC) in the literature.

During the controller design procedure complexity issues can occur that can inhibit the derivation of the controller, or the complexity of the resulting controller is so high that it is impossible to handle in real world operation. The TPDC framework offers trade-off techniques that help us to control the model complexity and approximation accuracy challenge. In this paper we derive TP models with different complexity of the TORA system, and design controllers that assures asymptotic stability.

In order to study the trade-off capability of TP model transformation we present a case study of the TORA system. We generate TP models with different complexity, and we derive controllers to each models. We analyze how the behavior of these models and the controllers' performance change compared to the exact TP model by reducing more and more the complexity.

The rest of the paper is organized as follows: Section II introduce the Tensor Product Distributed Compensation based controller design framework. Section III at first describes the TORA system, and discuss the goals and the specifications of the controller. Then, the different TP models are given, and through simulations the designed controllers are analyzed and compared. Finally, Section IV concludes the results.

II. Tensor Product Distributed Compensation (TPDC) based Controller Design Framework

Consider the following parameter-varying state-space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t), \quad (1)$$

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{D}(\mathbf{p}(t))\mathbf{u}(t),$$

with input $\mathbf{u}(t)$, output $\mathbf{y}(t)$ and state vector $\mathbf{x}(t)$. The system matrix

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t)) \\ \mathbf{C}(\mathbf{p}(t)) & \mathbf{D}(\mathbf{p}(t)) \end{pmatrix} \in \mathbb{R}^{O\times I} \quad (2)$$

is a parameter-varying object, where $\mathbf{p}(t) \in \Omega$ is time varying N-dimensional parameter vector, and is an element of the closed hypercube $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$. The parameter $\mathbf{p}(t)$ can also include some elements of $\mathbf{x}(t)$.

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The TP model transformation starts with the given LPV model (1) and results in the TP model representation

$$\begin{pmatrix}\dot{x}\\ y\end{pmatrix} \underset{\varepsilon}{\approx} \mathcal{S} \bigotimes_{n=1}^{N} \mathbf{w}_n(p_n) \begin{pmatrix}x\\ u\end{pmatrix} \qquad (3)$$

that can always be transformed to the polytopic form:

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{pmatrix} \underset{\varepsilon}{\approx} \sum_{r=1}^{R} w_r(\mathbf{p}(t))\mathbf{S}_r \begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} \qquad (4)$$

where

$$\varepsilon = \left(\left\| \mathbf{S}(\mathbf{p}(t)) - \sum_{r=1}^{R} w_r(\mathbf{p}(t))\mathbf{S}_r \right\|_{L_2} \right)^2 \leq \sum_k \sigma_k^2. \qquad (5)$$

Here, ε symbolizes the bounded approximation error, σ_k are the singular values discarded during the trade-off of the TP model transformation [5], [7], and $w_r(\mathbf{p}(t)) \in [0,1]$ are the coefficient functions. For further details about TP model transformation we refer to [4], [5], [8].

The TP model transformation ensures the convexity of the convex combination of the LTI systems as follows:

Definition 1: The model (4) is convex if:

$$\forall r \in [1,R], \mathbf{p}(t) \quad : \quad w_r(\mathbf{p}(t)) \in [0,1] \qquad (6)$$

$$\forall r \in [1,R], \mathbf{p}(t) \quad : \quad \sum_{i=1}^{I_n} w_r(\mathbf{p}(t)) = 1 \qquad (7)$$

This simply means that $\mathbf{S}(\mathbf{p}(t))$ is within the convex hull of the LTI vertex systems $\mathbf{S}_r$ for any $\mathbf{p}(t) \in \Omega$.

$\mathbf{S}(\mathbf{p}(t))$ has a finite element TP model representation in many cases ($\varepsilon = 0$ in (4)). In this case we say that the TP model is exact. However, exact finite element TP model representation does not exist in general ($\varepsilon > 0$ in (4)), see Ref. [9]. In this case $\varepsilon \to 0$, when the number of the LTI systems involved in the TP model goes to ∞.

The TP model transformation helps the trade-off between the complexity of the model, namely the number of LTI vertex systems, and the modeling accuracy, denoted by ε in Equation (5) [4]. The TP model transformation offers options to generate different types of the weighting functions $w(\cdot)$ to a given specification.

One class of the LMI based control design methods, the Parallel Distributed Compensation (PDC) framework was introduced by Tanaka and Wang [6]. The PDC design framework determines one LTI feedback gain to each LTI vertex system of a given TP model. The inputs of the framework are the LTI vertex systems $\mathbf{S}_r$, and the results are the LTI vertex gains $\mathbf{F}_r$ of the controller. These gains $\mathbf{F}_r$ are obtained from a feasible solution of the LMI based stability theorems. After having the $\mathbf{F}_r$, the control value $\mathbf{u}(t)$ is determined by the help of the same TP model structure used in (4):

$$\mathbf{u}(t) = -\left(\sum_{r=1}^{R} w_r(\mathbf{p}(t))\mathbf{F}_r\right)\mathbf{x}(t). \qquad (8)$$

The LMI theorems, to be solved under the PDC framework, are selected according to the stability criteria and the desired control performance. For instance, the speed of response, constraints on the state vector or on the control value can be considered via properly selected LMI based stability theorems.

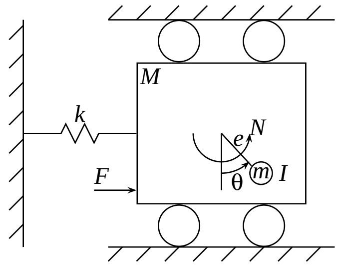

Fig. 1. TORA system

III. Complexity and approximation trade-off in the control of the TORA system

This section is devoted to show through the case study of the TORA system the approximation trade-off capabilities of the TP model transformation. Besides the exact TP model, we also generate complexity relaxed TP models. For each TP model a corresponding controller is designed. We analyze how the complexity relaxation changes the behavior of the TP models, and influence the controller performances. At the end of the section we make a comprehensive summary and comparison.

The study is conducted through a state feedback control design for the Translational Oscillations with an Eccentric Rotational Proof Mass Actuator (TORA) System, which was originally studied as a simplified model of a dual-spin spacecraft with mass imbalance to investigate the resonance capture phenomenon [10], [11]. The same plant was later studied involving the rotational proof-mass actuator for feedback stabilization of translational motion [12], [13]. The TORA system is also considered as a fourth-order benchmark problem [14]–[16]. The *International Journal of Robust and Nonlinear Control* published a series of studies about the control issue of the TORA system in Volume 8 in 1998 [17].

A. Nomenclature

- M = mass of cart
- k = linear spring stiffness
- m = mass of the proof-mass actuator
- I = moment of inertia of the proof-mass actuator
- e = distance between the rotation point and the center of the proof mass
- N = control torque applied to the proof mass
- q = translational position of the cart
- θ = angular position of the rotational proof mass

B. Equations of motion

The TORA system is shown in Figure 1 with the notation defined above. The oscillation consists of a cart of mass M connected to a fixed wall by a linear spring of stiffness k. The cart is constrained to have one-dimensional travel in the horizontal plane. The rotating proof-mass actuator is attached to the cart. The control torque is applied to the proof mass. The $\theta = 0^\circ$ is perpendicular to the motion of the cart, while $\theta = 90^\circ$ is aligned with the positive q direction. The equations of motion are given by [17]:

$$(M+m)\ddot{q} + kq = -me(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \qquad (9)$$

$$(I+me^2)\ddot{\theta} = -me\ddot{q}\cos\theta + N \tag{10}$$

with the normalization [12]:

$$\xi \simeq \sqrt{\frac{M+m}{I+me^2}}q \quad \tau \simeq \sqrt{\frac{k}{M+m}}t \tag{11}$$

$$u \simeq \frac{M+m}{k(I+me^2)}N \tag{12}$$

the equations of motion become

$$\ddot{\xi} + \xi = \varepsilon\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right) \tag{13}$$

$$\ddot{\theta} = -\varepsilon\ddot{\xi}\cos\theta + u \tag{14}$$

where ξ is the normalized cart position, and u is the non-dimensional control torque. τ is the normalized time whereupon the differentiation is understood. The ε is the coupling between the rotational and the translational motions:

$$\varepsilon \simeq \frac{me}{\sqrt{(I+me^2)(M+m)}}. \tag{15}$$

The above equations can be given in the state-space model form

$$\dot{x}(t) = f(x(t)) + g(x(t))u, \tag{16}$$

$$y(t) = c(x(t)),$$

where

$$f(x(t)) = \begin{pmatrix} x_2(t) \\ \frac{-x_1(t)+\varepsilon x_4(t)^2\sin x_3(t)}{1-\varepsilon^2\cos^2 x_3(t)} \\ x_4(t) \\ \frac{\varepsilon\cos x_3(t)(x_1(t)-\varepsilon x_4(t)^2\sin x_3(t))}{1-\varepsilon^2\cos^2 x_3(t)} \end{pmatrix}, \quad g(x(t)) = \begin{pmatrix} 0 \\ \frac{-\varepsilon\cos x_3(t)}{1-\varepsilon^2\cos^2 x_3(t)} \\ 0 \\ \frac{1}{1-\varepsilon^2\cos^2 x_3(t)} \end{pmatrix}, \tag{17}$$

$$c(x(t)) = \begin{pmatrix} x_1(t) & 0 & 0 & 0 \\ 0 & 0 & x_3(t) & 0 \end{pmatrix}.$$

and $x(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{pmatrix}^T = \begin{pmatrix} \xi & \dot{\xi} & \theta & \dot{\theta} \end{pmatrix}^T$. Let us write the above equation in the typical form of linear parameter varying state-space model as

$$\dot{x}(t) = \mathbf{S}(\mathbf{p}(t))\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \quad y(t) = \mathbf{C}x(t), \tag{18}$$

where system matrix $\mathbf{S}(\mathbf{p}(t))$ contains:

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t)) \end{pmatrix}$$

and $\mathbf{p}(t) = \begin{pmatrix} x_3(t) & x_4(t) \end{pmatrix} \in \mathbf{\Omega}$ is time varying 2-dimensional parameter vector, so as

$$\mathbf{A}(x_3(t), x_4(t)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{1-\varepsilon^2\cos^2 x_3(t)} & 0 & 0 & \frac{\varepsilon x_4(t)\sin x_3(t)}{1-\varepsilon^2\cos^2 x_3(t)} \\ 0 & 0 & 0 & 1 \\ \frac{\varepsilon\cos x_3(t)}{1-\varepsilon^2\cos^2 x_3(t)} & 0 & 0 & \frac{-x_4(t)\varepsilon^2\cos x_3(t)\sin x_3(t)}{1-\varepsilon^2\cos^2 x_3(t)} \end{pmatrix} \tag{19}$$

$$\mathbf{B}(x_3(t)) = g(x(t)) \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The laboratory version of the TORA system is described in [15]. The nominal configuration of this version is given in Table I.

TABLE I
PARAMETERS OF THE TORA SYSTEM

Description	Parameter	Value	Units
Cart mass	M	1.3608	kg
Arm mass	m	0.096	kg
Arm eccentricity	e	0.0592	m
Arm inertia	I	0.0002175	kg m^2
Spring stiffness	k	186.3	N/m
Coupling parameter	ε	0.200	—

C. Different complexity relaxed TP models generated by the TP model transformation

We execute the TP model transformation on the LPV model (18) of the TORA. As a first step of the TP model transformation we have to define the transformation space Ω. By considering the behavior of the TORA system and the controller design specifications described in the following subsection, we define the transformation space as $\Omega = [-a,a] \times [-a,a]$ ($x_3(t) \in [-a,a]$ and $x_4(t) \in [-a,a]$), where $a = \frac{45}{180}\pi$rad (note that these intervals can be arbitrarily defined). The TP model transformation starts with the discretization over a rectangular grid. Let the density of the discretization grid be 101×101 on $(x_3(t) \in [-a,a]) \times (x_4(t) \in [-a,a])$. The result of the TP model transformation shows that the rank of $\mathbf{S}(p)$ in the dimension of x_3 is 4, whilst in the dimension of x_4 is 2. The singular values in each dimensions are the following: $\sigma_{1,1} = 251.62, \sigma_{1,2} = 5.7833, \sigma_{1,3} = 2.8396, \sigma_{1,4} = 0.030969$; and $\sigma_{1,2} = 251.63, \sigma_{2,2} = 5.7833$.

TP MODEL 0: As the TP model transformation resulted in finite number of singular values, the TORA system can be exactly given as the combination of $4 \times 2 = 8$ LTI systems:

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{r=1}^{8} w_r(x_3(t), x_4(t))\left(\mathbf{A}_{i,j}\mathbf{x}(\mathbf{t}) + \mathbf{B}(\mathbf{t})_{i,j}u(t)\right). \tag{20}$$

The close to NO type weighting functions that define the tight convex hull of the LPV model is depicted in Figure 2. We consider the TP MODEL 0 as the reference model in the followings.

TP MODEL 1: The complexity of the TP model can be reduced in the dimension of x_3 by discarding some singular values. Note that in the dimension of x_4 the reduction is not possible since convexity (Definition 1) requires at least two weighting functions. Hence, let us keep three largest singular values of dimension x_3. It results an approximation of the TORA system that is composed of $3 \times 2 = 6$ LTI systems. Figure 3 shows the weighting functions of the tight convex hull of the reduced LPV model.

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{r=1}^{6} w_r(x_3(t), x_4(t))\left(\mathbf{A}_{i,j}\mathbf{x}(\mathbf{t}) + \mathbf{B}(\mathbf{t})_{i,j}u(t)\right). \tag{21}$$

Equation (5) gives only an error bound for the approximation error. In order to measure the actual modeling approximation error, the difference of the analytical model and TP models, in terms of L_2 error, were calculated over 10 000 random sample points.

The upper bound for TP MODEL 1 is the sum of discarded singular values that is $\sigma_{1,4} = 0.030969$. By numerical checking the maximal measured L_2 approximation error is 0.0007.

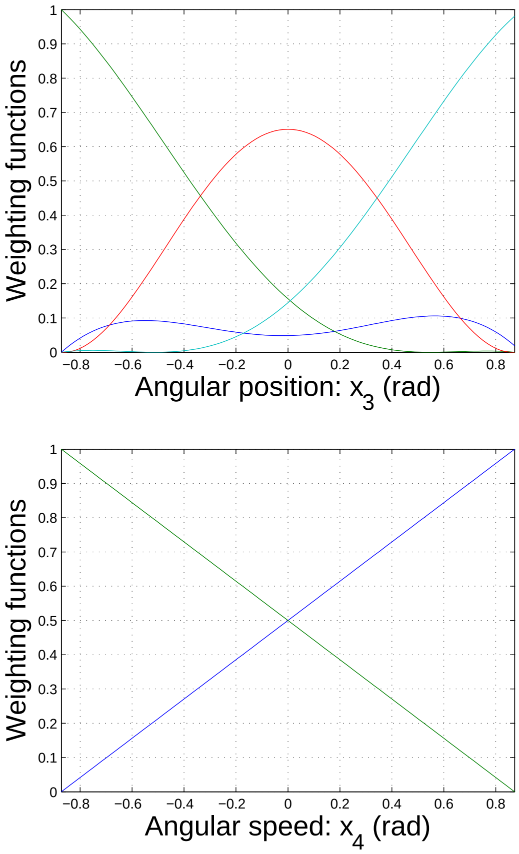

Fig. 2. Close to NO type weighting functions of the exact TP model

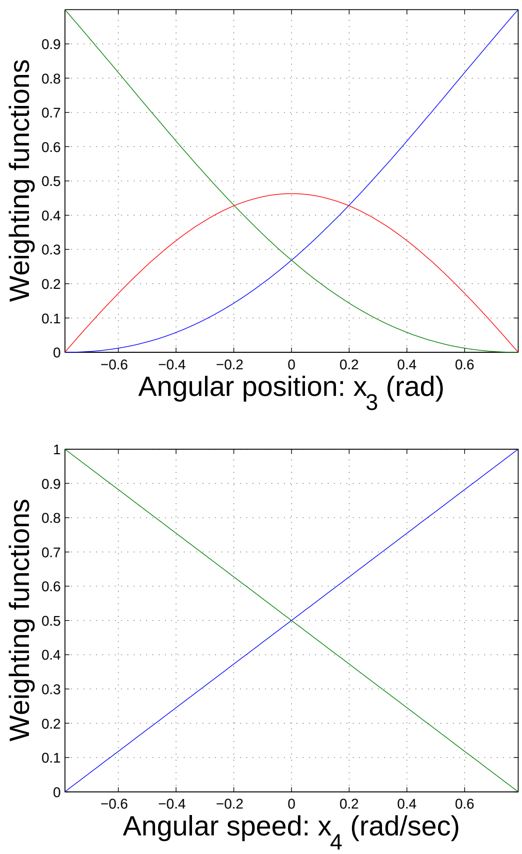

Fig. 3. Close to NO type weighting functions of the reduced TP model of 6 LTI systems

TP MODEL 2: In the dimension of x_3 further reduction is possible. By discarding the two smallest singular values, namely $\sigma_{1,3}$ and $\sigma_{1,4}$, the most reduced TP model of the TORA system realizes. The resulting system equation is:

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{r=1}^{6} w_r(x_3(t), x_4(t))\,(\mathbf{A}_{i,j}\mathbf{x}(\mathbf{t}) + \mathbf{B}_{i,j}u(t))\,. \tag{22}$$

In this case the upper bound of the approximation error is 2.8699, while the maximal measured error is 0.3033. In Figure 4 the resulting weight functions are illustrated.

COMPARISON OF THE RESULTING TP MODELS: The results, which is summarized in Table II, show that the estimated error bound from the singular values are much worse than the actual approximation error and the complexity of the model can be drastically reduced without causing unacceptable approximation error. However, even a slight approximation error can cause a change in an important aspect of behavior, so it should be checked and reduction should be done with precaution.

D. Derivation of controllers

For the present controller design, we apply the widely adapted design specifications detailed on [17, page 309] and in [18]–[20]. These specifications for the TORA system can be summarized as follows:

Design a controller that satisfies the following criteria:

- The closed-loop system exhibits good settling behavior for a class of initial conditions.
- The closed-loop system is stable (in this paper we aim at achieving asymptotic stability).
- The physical configuration of the system necessities the constraint $|q| \leq 0.025\,\mathrm{m}$.
- The control value is limited by $N \leq 0.100\,\mathrm{Nm}$, although somewhat higher torques can be tolerated for short periods.

Considering these specifications, we derive controllers by applying the TP model and the LMI theorems. Having the solution of the LMIs the feedback gains are computed by Equation (25), and the control value is computed by Equation (8). In the present case it is:

$$u(t) = -\left(\sum_{r=1}^{R} w_r(x_3(t), x_4(t))\mathbf{F}_r\right)x(t),$$

where R is the number of LTI systems of the applied TP model.

The following LMI systems are used to design the controller. The derivations and the proofs of these theorems are fully detailed in [6].

Theorem 1 (Asymptotic stability): TP model (4) with control value (8) is asymptotically stable if there exists $\mathbf{X} > 0$ and $\mathbf{M}_r$ satisfying equations

$$-\mathbf{X}\mathbf{A}_r^T - \mathbf{A}_r\mathbf{X} + \mathbf{M}_r^T\mathbf{B}_r^T + \mathbf{B}_r\mathbf{M}_r > \mathbf{0} \tag{23}$$

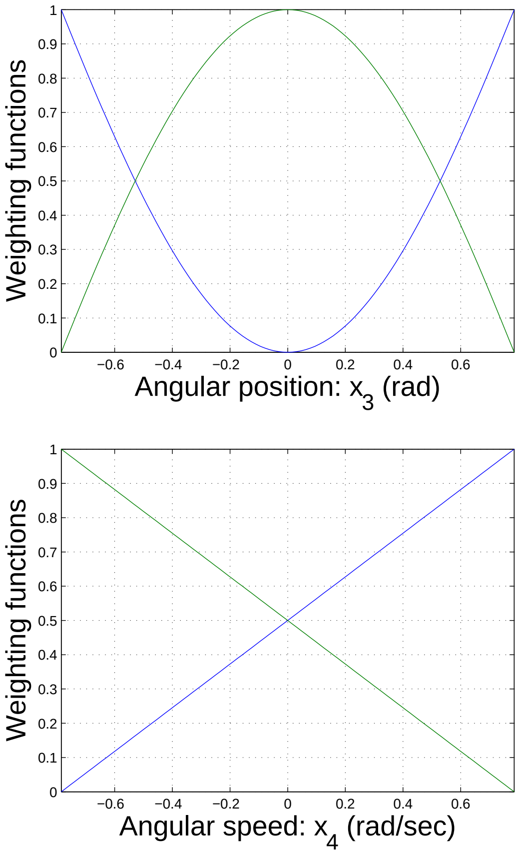

Fig. 4. Close to NO type weighting functions of the reduced TP model of 4 LTI systems

for all r and

$$-\mathbf{X}\mathbf{A}_r^T - \mathbf{A}_r\mathbf{X} - \mathbf{X}\mathbf{A}_s^T - \mathbf{A}_s\mathbf{X} + \tag{24}$$

$$+\mathbf{M}_s^T\mathbf{B}_r^T + \mathbf{B}_r\mathbf{M}_s + \mathbf{M}_r^T\mathbf{B}_s^T + \mathbf{B}_s\mathbf{M}_r \geq \mathbf{0}.$$

for $r < s \leq R$, except for the pairs (r,s) such that $w_r(\mathbf{p}(t))w_s(\mathbf{p}(t)) = 0, \forall \mathbf{p}(t)$.

Theorem 2 (Constraint on the control value): Assume that $\|\mathbf{x}(0)\| \leq \phi$, where $\mathbf{x}(0)$ is unknown, but the upper bound ϕ is known. The constraint $\|\mathbf{u}(t)\|_2 \leq \mu$ is enforced at all times $t \geq 0$ if the LMIs

$$\phi^2\mathbf{I} \leq \mathbf{X}$$

$$\begin{pmatrix} \mathbf{X} & \mathbf{M}_i^T \\ \mathbf{M}_i & \mu^2\mathbf{I} \end{pmatrix} \geq \mathbf{0}$$

hold.

Theorem 3 (Constraint on the output): Assume that $\|\mathbf{x}(0)\| \leq \phi$, where $\mathbf{x}(0)$ is unknown, but the upper bound ϕ is known. The constraint $\|\mathbf{y}(t)\|_2 \leq \lambda$ is enforced at all times $t \geq 0$ if the LMIs

$$\phi^2\mathbf{I} \leq \mathbf{X}$$

$$\begin{pmatrix} \mathbf{X} & \mathbf{X}\mathbf{C}_i^T \\ \mathbf{C}_i\mathbf{X} & \lambda^2\mathbf{I} \end{pmatrix} \geq \mathbf{0}$$

hold.

The feedback gains are determined form the solutions $\mathbf{X}$ and $\mathbf{M}_r$ as

$$\mathbf{F}_r = \mathbf{M}_r\mathbf{X}^{-1}. \tag{25}$$

We compose a joint LMI system of Theorem 1–3 to guarantee the stability issues and constraints. The feasible solution of this joint LMI system can be easily computed by an LMI solver, *e.g.* the one included in MATLAB Robust Control Toolbox.

CONTROLLER 0: This controller is designed for TP MODEL 0. This requires the solution of 46 LMI equations. The feedback gains of the controller are:

$$\begin{aligned}
\mathbf{F}_1 &= \begin{pmatrix} -0.91209 & 0.038408 & 0.69945 & 1.6801 \end{pmatrix} \\
\mathbf{F}_2 &= \begin{pmatrix} -1.0416 & -0.10353 & 0.58048 & 1.585 \end{pmatrix} \\
\mathbf{F}_3 &= \begin{pmatrix} -0.63713 & -0.30129 & 0.38614 & 1.2927 \end{pmatrix} \\
\mathbf{F}_4 &= \begin{pmatrix} -0.8723 & 0.17153 & 0.80205 & 1.7349 \end{pmatrix} \\
\mathbf{F}_5 &= \begin{pmatrix} -0.92539 & 0.046389 & 0.70622 & 1.6878 \end{pmatrix} \\
\mathbf{F}_6 &= \begin{pmatrix} -0.63713 & -0.30129 & 0.38614 & 1.2927 \end{pmatrix} \\
\mathbf{F}_7 &= \begin{pmatrix} -1.0416 & -0.10353 & 0.58048 & 1.585 \end{pmatrix} \\
\mathbf{F}_8 &= \begin{pmatrix} -0.84626 & 0.15441 & 0.78774 & 1.7191 \end{pmatrix}
\end{aligned}$$

The control value $u(t)$ is computed by the following equation:

$$u(t) = -\left(\sum_{r=1}^{8} w_r(x_3(t), x_4(t))\mathbf{F}_r\right) x(t),$$

where w_rs are the weight functions of the corresponding TP model. We consider this controller as the reference controller, and the response of the rest of the controllers are compared to this.

CONTROLLER 1: The LMI system of TP MODEL 1 consists of 29 LMIs. The feedback gains of the controller, that is derived from the feasible solution of LMI system, is the following:

$$\begin{aligned}
\mathbf{F}_1 &= \begin{pmatrix} -0.97671 & -0.12733 & 0.57912 & 1.4513 \end{pmatrix} \\
\mathbf{F}_2 &= \begin{pmatrix} -0.60715 & -0.31922 & 0.38945 & 1.2174 \end{pmatrix} \\
\mathbf{F}_3 &= \begin{pmatrix} -0.82162 & 0.079923 & 0.74673 & 1.5698 \end{pmatrix} \\
\mathbf{F}_4 &= \begin{pmatrix} -0.60715 & -0.31922 & 0.38945 & 1.2174 \end{pmatrix} \\
\mathbf{F}_5 &= \begin{pmatrix} -0.97671 & -0.12733 & 0.57912 & 1.4513 \end{pmatrix} \\
\mathbf{F}_6 &= \begin{pmatrix} -0.82162 & 0.079922 & 0.74673 & 1.5698 \end{pmatrix}
\end{aligned}$$

The control value $u(t)$ is computed by the following equation:

$$u(t) = -\left(\sum_{r=1}^{6} w_r(x_3(t), x_4(t))\mathbf{F}_r\right) x(t),$$

where w_rs are the weight functions of TP MODEL 1.

CONTROLLER 2: We applied the same LMI system to TP MODEL 2 that results a system of 16 LMI equations. The feedback gains of the controller, that is derived from the feasible solution of LMI system, is the following:

$$
\begin{aligned}
\mathbf{F}_1 &= \begin{pmatrix} -0.36782 & -0.289 & 0.41269 & 0.72065 \end{pmatrix}\\
\mathbf{F}_2 &= \begin{pmatrix} -0.33796 & -0.19934 & 0.46307 & 0.78012 \end{pmatrix}\\
\mathbf{F}_3 &= \begin{pmatrix} -0.36782 & -0.289 & 0.41269 & 0.72065 \end{pmatrix}\\
\mathbf{F}_4 &= \begin{pmatrix} -0.33796 & -0.19934 & 0.46307 & 0.78012 \end{pmatrix}
\end{aligned}
$$

The control value $u(t)$ is computed by the following equation:

$$
u(t) = -\left(\sum_{r=1}^{4} w_r(x_3(t), x_4(t))\mathbf{F}_r\right) x(t),
$$

where w_rs are the weight functions of TP MODEL 2.

E. Simulation results and comparison of derived controllers

The simulation results were obtained using the `ode45` command of MATLAB Simulink. The system's initial configuration was $x(0) = \begin{pmatrix} 0.023\,\mathrm{m} & 0 & 0 & 0 \end{pmatrix}$. The results of the three controllers are shown and are plotted together for better visualization in Figure 5. Figure 6 shows some parts of the simulation results magnified in order to highlight the differences. We can see on the figures, there are only slight differences between the responses of the different controllers, practically we can say that the results are identical despite of the applied reduction during the TP model transformation. The main reason behind this fact is that the strength of influence of the LTI models is proportional to the magnitude of the singular values. Therefore, the magnitude of differences between the designed controllers is also strongly related to the magnitude of the singular values. For illustration, we should analyze carefully the responses of the controller of the exact model (indicated with "CTRL 0" in the figures), and the controller CONTROLLER 1. The difference is so small, because the difference of the exact TP model, TP MODEL 0 and the TP MODEL 1 from which the controllers were derived is also really small. The contribution of the neglected LTIs to the TP model is proportional to the ratios of the singular values, and $\sigma_{1,4}$ has only an effect of 0.04%. If we analyze the response of controller CONTROLLER 2, we can see more larger differences, because the contribution of $\sigma_{1,3}$ is around 1.09%, thus together with $\sigma_{1,4}$ it is around 1.13% in total, that is more significant.

Another important issue concerning these results is that in order to derive CONTROLLER 0, an LMI system of 46 LMIs has to be solved, whilst 16 LMIs can describe CONTROLLER 2, the controller of the most reduced TP model that is a 65% of reduction. The TORA system is a simple model, the number of LMIs is moderate, but by defining more constraints, applying more complex controller specifications, such as decay rate control, observer design, etc., and if the TP model of the system consists of more LTI systems, the number of LMIs can easily explode to such a manner that is difficult to handle [21].

Table II shows a comprehensive chart on the approximation trade-off of the TORA system.

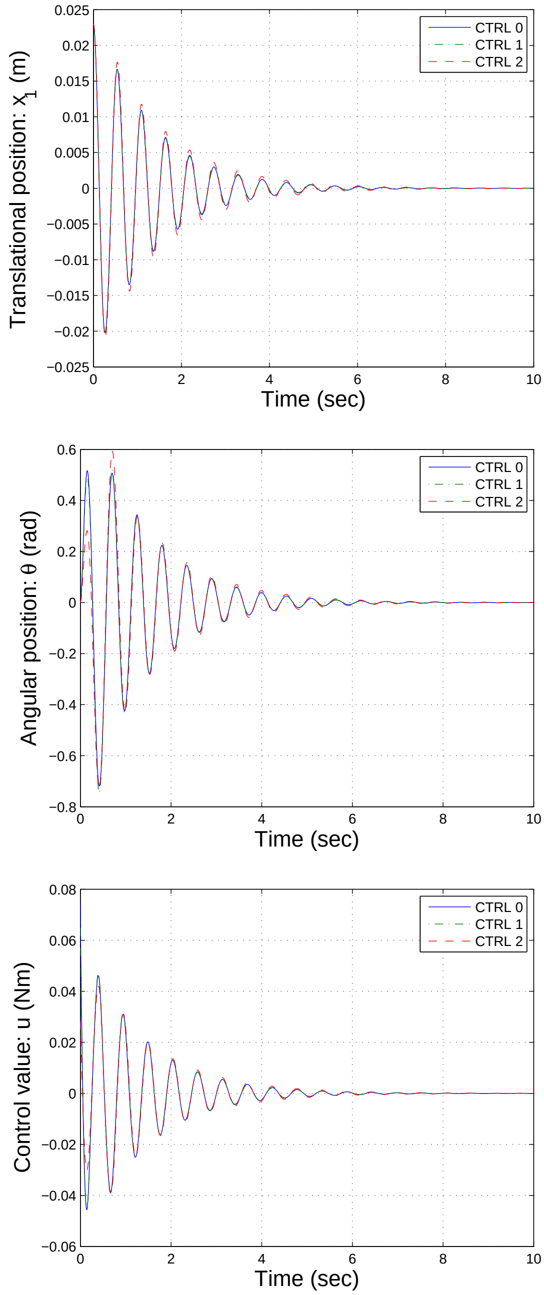

Fig. 5. Asymptotic stability controller design of exact and reduced TP models

IV. CONCLUSION

The paper presents a study how the TPDC controller design framework can handle the trade-off between approximation accuracy and model complexity through the case study of the TORA system. The proposed framework is shown to be an efficient tool for complexity reduction. The simulation proved that there is no significant difference in control response between the controllers derived from the reference model and the reduced models whilst major loss in model size and cut in computational necessity is achieved. As a matter of fact we should note that the stability issues of the original model are not fully guaranteed by the controllers derived from the

TABLE II
SUMMARY OF APPROXIMATION TRADE-OFF OF THE TORA SYSTEM

Number of singular values kept	Number of LTIs	Reduction ratio of model transformation	Upper-bound of estimated error	Measured maximal L_2 error	Number of LMIs of the controller	Reduction ration of the number of LMIs
4	8	0%	0	10^{-12}	46	0%
3	6	25%	0.0309	0.0007	29	27%
2	4	50%	2.8699	0.3033	16	65%

reduced TP models.

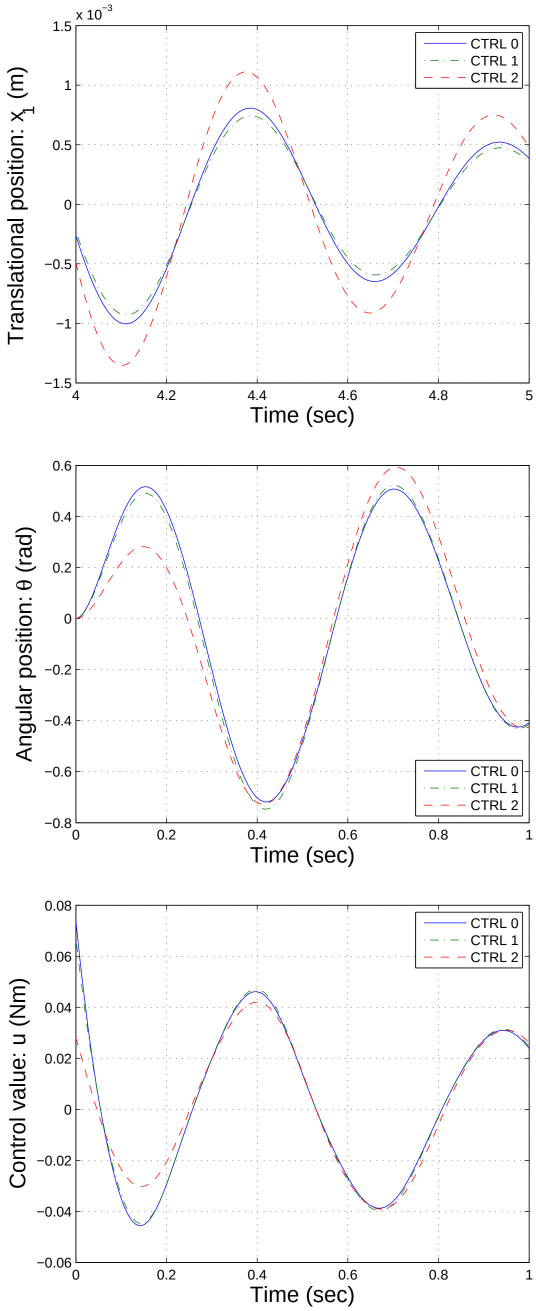

Fig. 6. Magnification of Figure 5 for emphasizing the differences of controllers

REFERENCES

[1] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, "Linear matrix inequalities in system and control theory," *Philadelphia PA:SIAM, ISBN 0-89871-334-X*, 1994.

[2] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox*. The MathWorks, Inc., 1995.

[3] C. W. Scherer and S. Weiland, *Linear Matrix Iequalities in Control*, ser. DISC course lecture notes, DOWNLOAD: http://www.cs.ele.tue.nl/SWeiland/lmid.pdf, 2000.

[4] P. Baranyi, "TP model transformation as a way to LMI based controller design," *IEEE Transaction on Industrial Electronics*, vol. 51, no. 2, pp. 387–400, April 2004.

[5] P. Baranyi, D. Tikk, Y. Yam, and R. J. Patton, "From differential equations to PDC controller design via numerical transformation," *Computers in Industry, Elsevier Science*, vol. 51, pp. 281–297, 2003.

[6] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis — A Linear Matrix Inequality Approach*. John Wiley and Sons, Inc., 2001.

[7] Y. Yam, P. Baranyi, and C. T. Yang, "Reduction of fuzzy rule base via singular value decomposition," *IEEE Transaction on Fuzzy Systems*, vol. 7, no. 2, pp. 120–132, 1999.

[8] P. Baranyi, "Tensor product model based control of 2-D aeroelastic system," *Journal of Guidance, Control, and Dynamics (in Press)*.

[9] D. Tikk, P. Baranyi, R. J. Patton, and J. Tar, "Approximation Capability of TP model forms," *Australian Journal of Intelligent Information Processing Systems*, vol. 8, no. 3, pp. 155–163, 2004.

[10] R. H. Rand, R. J. Kinesey, and D. L. Mingori, "Dynamics of spinup through resonance," *International Journal Non-linear Mechanics*, vol. 27, pp. 489–502, 1992.

[11] M. Jankovic, D. Fontanie, and P. V. Kokotovic, "Tora example: cascade- and passivity-based control designs," *IEEE Transaction on Control System Technologies*, vol. 4, pp. 292–297, 1996.

[12] C. J. Wan, D. S. Bernstein, and V. T. Coppola, "Global stabilisation of the oscillating eccentric rotor," *Nonlinear Dynamics*, vol. 10, pp. 49–62, 1996.

[13] R. Bupp, V. T. Coppola, and D. S. Bernstein, "Vibration suppression of multi-modal translational motion using a rotational actuator," in *Proc. of the IEEE Int. Decision and Control*, Orlando, FL, 1994, pp. 4030–4034.

[14] B. Wie and D. S. Bernstein, "Benchmark problems in robust control design," *Journal of Guidance, Control and Dynamics*, vol. 15, pp. 1057–1059, 1992.

[15] R. Bupp, D. S. Bernstein, and V. T. Coppola, "A benchmark problem for nonlinear control design: problem statement, experimental testbed, and passive nonlinear compensation," in *Proc. American Control Conference*, Seattle, WA, 1995, pp. 4363–4367.

[16] ——, "A benchmark problem for nonlinear control design," *International Journal of Robust and Nonlinear Control*, vol. 8, pp. 307–310, 1998.

[17] Special, "This special issue presents 9 papers dealing with the control of the tora system," *International Journal of Robust and Nonlinear Control*, vol. 8, pp. 305–457, 1998.

[18] G. Tadmor, "Dissipative design, lossless dynamics, and the nonlinear tora benchmark example," *IEEE Transaction on Control System Technology*, vol. 9, no. 2, pp. 391–398, 2001.

[19] K. Tanaka, T. Taniguchi, and H. O. Wang, "Model based fuzzy control of TORA system: Fuzzy regulator amd fuzzy observer design via LMIs that represent decay rate, disturbance rjection, robustness, optimality," in *Proc. of Int. IEEE Conference on Fuzzy Systems*, Anchorage, Alaska, USA, 1998, pp. 313–318.

[20] G. Escobar, R. Ortega, and H. Sira-Ramirez, “Output-feedback global stabilization of a nonlinear benchmark system using a saturated passivity-based controller,” *IEEE Transaction on control system technology*, vol. 7, no. 2, pp. 289–293, 1999.

[21] D. Tikk, P. Baranyi, R. J. Patton, and J. K. Tar, “Approximation capability of TP model forms,” *Australian Journal of Intelligent Information Processing Systems*, vol. 8, no. 3, pp. 155–163, 2004.