DESYCHRONIZATION OF ONE-PARAMETER FAMILIES OF STABLE VECTOR FIELDS

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The results reported here arised from the long term motivation of finding new therapeutic strategies. Living systems are open systems exchanging matter, energy and information with their environment, but survival is strongly subordinated to the constancy of internal parameters. Those homeostatic phenomena are illustrated by isothermy or maintenance of sugar concentration in blood, or regulation of absorption of toxic mineral micronutrients like copper as examples. On another part it is a constatation that many metabolic pathways, glycolysis for instance or amino acid synthesis, tend to reach a stable equilibrium. Given that the metabolism is a very stable dynamical system, our motivation was to find and describe a way to destabilize this system.

Let us recall what is master-slave synchronization on an example. The celebrated Lorenz dynamical system:

$$(L)\left\{\begin{array}{l}\dot{x} \,=\, \sigma(y-x)\\ \dot{y} \,=\, rx-y-xz\\ \dot{z} \,=\, xy-bz\end{array}\right.$$

is known to exhibit chaotic behaviour for some values of σ, r and b. Even though, it was noticed at first by Pecora and Carroll that given any solution $(x(t), y(t), z(t))$, the reduced (non-autonomous) system of two variables:

$$(L')\left\{\begin{array}{l}\dot{Y} \,=\, rx(t)-Y-x(t)Z\\ \dot{Z} \,=\, x(t)Y-bZ\end{array}\right.$$

has all its trajectories $(Y(t), Z(t))$ asymptotically converging to $(y(t), z(t))$, that is:

$$\lim_{t\to+\infty}||Y(t)-y(t)||=0 \qquad \lim_{t\to+\infty}||Z(t)-z(t)||=0.$$

This phenomenon was called *master-slave synchronization*. Pecora and Carroll noticed an even more surprising fact: when *any* function is substituted to $x(t)$

inside the equations of the reduced system (L'), any two solutions $(Y_1(t), Z_1(t))$ and $(Y_2(t), Z_2(t))$ of (L') would eventually converge one toward the other.

Taking this example from the opposite view, suppose that we had been given the one-parameter family of vectors fields:

$$(L_\lambda) \begin{cases} \dot{Y} = r\lambda - Y - \lambda Z \\ \dot{Z} = \lambda Y - bZ \end{cases}$$

For each λ, the system (L_λ) has a stable equilibrium point at $(\frac{rb\lambda}{\lambda^2+b}, \frac{r\lambda^2}{\lambda^2+b})$. Then we can wonder whether it is possible to make the system chaotic or at least break the asymptotic stability of the system by "moving" λ. In the above particular example, we know the answer:

- if we "shake" the parameter by an external action then the answer is NO. In effect, this would mean to replace λ by a function and since the system is absolutely MS-synchronizable, all solutions would have the same future;
- if we let the system move the parameter by itself, it has a chance of success, that is if we add a differential equation for λ the enlarged system can show chaos. A convenient differential equation is evidently the equation for x in the Lorenz system with the good choice of parameters r, a, b.

Ironically, we could say that (L_λ) has a property of self-disorganization. This concept qualifies the ability of a parametrized system to brake its expected behaviour by acting on external parameters. It can be used whenever there is an interest in destabilizing a system.

Here, we adress the question in the large: given a one-parameter family of vector fields each one having a globally stable equilibrium point, in which conditions does it possess the property of self-disorganization? Our answer reads:

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval, let $U \subset \mathbb{R}^n$ be an open set and let $\lambda \in I \mapsto F_\lambda$ be a C^k family of C^k vector fields, $k \geq 1$ or $k = \infty$ defined on U. Assume that, for each $\lambda \in I$ there exist $x^*_\lambda \in U$ which is a hyperbolic attracting singularity for the differential equation:*

$$\frac{dx}{dt} = F_\lambda(x).$$

that is $F_\lambda(x^*_\lambda) = 0$ and $D_x F_\lambda(x^*_\lambda)$ has all its eigenvalues with negative real parts. Assume furthermore that (i) there exist only one real eigenvalue with maximum negative real part or only one pair of complex conjugate eigenvalues with maximum real part, and (ii) there exists $\lambda_0 \in I$ such that $\partial_\lambda F_\lambda(x^*_\lambda)_{|\lambda=\lambda_0} \neq 0$.
Then there exists C^∞-map $g : I \times U \to \mathbb{R}$ such that the vector field $G(\lambda, x) = (g(\lambda, x), F_\lambda(x))$ is chaotic.

Remark 1.

(1) The hypothesis (i) is generic. In particular, if two eigenvalues which are not complex conjugate have the same real part, a small perturbation of F would not keep this property.
(2) The hypothesis (ii) made on $\partial_\lambda F_\lambda(x^*_\lambda)_{|\lambda=\lambda_0}$ is a transversality condition. It expresses the fact that the family of vector fields has a order one dependence on the parameter.

(3) By chaotic, we mean that there exists a first return map of the vector field G which has horseshoes, and by consequence, positive topological entropy. In particular, the orbits of G show sensibility to initial conditions.
(4) It is important for applications to note that the chaotic behaviour does not only concern what was called in the initial system the parameter, but also the initial variables of the system.
(5) More can be said from the proof of Theorem 1: there exist l_0 and l_1 in I and neighborhoods V_0 and V_1 of $x^*_{l_0}$ and $x^*_{l_1}$ respectively, such that the orbit of certain points in $V_0 \cup V_1$ visits infinitly often V_0 and V_1 alternatively. This oscillating behaviour in contrast with the asymptotic stability interestingly illustrates self-disorganization. This property together with sensibility to initial conditions might be exploited in the theme of drug design mentioned earlier.

The result has been published in *Nonlinearity* (19 (2006) 261-276).

In the presentation I will state the result, give an idea of the proof based on the construction of a homoclinic loop and give numerical illustrations. Finally, I will adress the question of making the theorem evolve into a concrete therapeutic device.