On the stability of a convective flow in a pipe generated by internal heat sources

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Introduction

Convective motions occur in fluids when there is a non-uniform temperature distribution across the fluid layer. As the temperature difference increases, the intensity of the convective motion also increases and the flow may become unstable. Many instability mechanisms are found to exist for non-isothermal flows. Much attention has been given in the literature to the study of convective motions caused by internal heat sources. Convective flows with internal heat generation play an important role in the solution of several problems of practical interest. Examples include Joule heating in an electrolyte or processes occurring in the Earth's mantle [1], [2]. Thermal convection in a fluid with internal heat sources is very important in the theory of thermal ignition [3]. Another important application is the miniaturization of electronic components [4], where the heat sources are situated along the boundary of a given region.

The stability of a convective motion in a vertical fluid layer with uniform heat sources is studied in [5], [6], where it is shown that at low Prandtl numbers the role of thermal perturbations is relatively small so that the energy is transported mainly from the base flow to the disturbances. At large Prandtl numbers the instability occurs in the form of thermal running waves.

Mathematical analysis

Consider the system of the Navier–Stokes equations in the Boussinesq approximation written in dimensionless form

$$\begin{aligned}
\frac{\partial v_r}{\partial t} + Gr\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) &= -\frac{\partial p}{\partial r} + \nabla^2 v_r \\
&\quad - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r^2}, \qquad (1)\\
\frac{\partial v_\theta}{\partial t} + Gr\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \nabla^2 v_\theta \\
&\quad + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}, \qquad (2)\\
\frac{\partial v_z}{\partial t} + Gr\left(v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} - \frac{v_\theta^2}{r}\right) &= -\frac{\partial p}{\partial z} + \nabla^2 v_r + T, \qquad (3)\\
\frac{\partial T}{\partial t} + Gr\left(v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) &= \frac{1}{Pr}\nabla^2 T + \frac{1}{Pr}, \qquad (4)\\
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} &= 0, \qquad (5)
\end{aligned}$$

where (r, θ, z) are cylindrical polar coordinates, t is time, v_r, v_θ and v_z are the velocity components, T is the temperature, p is the pressure, the measures of length, time, velocity, temperature and pressure are R, R^2/ν, $g\beta QR^4/(\nu\kappa\rho c_p)$, $QR^2/(\rho\kappa c_p)$ and $g\beta QR^3/(\kappa c_p)$, respectively. The Grashof number Gr and the Prandtl number Pr are defined as follows: $Gr = g\beta QR^5/(\nu^2\kappa\rho c_p)$, $Pr = \nu/\kappa$.

The system (1)–(5) admits the solution of the following form

$$\mathbf{V} = (0, 0, W_0(r)), \quad T = T_0(r), \quad p = P_0(z). \qquad (6)$$

Substituting (6) into (1)–(5) we obtain the system

$$-\frac{dP_0}{dz} + \frac{d^2 W_0}{dr^2} + \frac{1}{r}\frac{dW_0}{dr} + T_0 = 0, \qquad (7)$$

$$\frac{d^2 T_0}{dr^2} + \frac{1}{r}\frac{dT_0}{dr} + 1 = 0. \qquad (8)$$

The boundary conditions are

$$W_0\Big|_{r=1} = 0, \quad T_0\Big|_{r=1} = 0. \qquad (9)$$

The functions $W_0(r)$ and $T_0(r)$ are bounded at $r = 0$. In addition, we assume that the pipe is closed so that the total fluid flux through the cross–section of the pipe is zero:

$$\int_0^1 rW_0(r)\,dr = 0. \tag{10}$$

The solution of (6)–(10) has the form

$$W_0(r) = \frac{1}{192}(1 - 4r^2 + 3r^4), \quad T_0 = \frac{1}{4}(1 - r^2), \quad P_0 = \frac{1}{6}z + \text{const}. \tag{11}$$

We consider the stability of the flow (11) by the method of normal perturbations. That is, the perturbations are sought in the form

$$\begin{array}{rcl} v_r(r, z, \varphi, t) &=& u(r)\,\exp\left(-\lambda t + in\varphi + ikz\right), \\ v_\varphi(r, z, \varphi, t) &=& v(r)\,\exp\left(-\lambda t + in\varphi + ikz\right), \\ v_z(r, z, \varphi, t) &=& w(r)\,\exp\left(-\lambda t + in\varphi + ikz\right), \\ p(r, z, \varphi, t) &=& q(r)\,\exp\left(-\lambda t + in\varphi + ikz\right), \\ T(r, z, \varphi, t) &=& \theta(r)\,\exp\left(-\lambda t + in\varphi + ikz\right) \end{array} \tag{12}$$

and the equations of motion (1)–(5) are linearized in the neighborhood of the flow (11). Here $u(r)$, $v(r)$, $w(r)$, $q(r)$ and $\theta(r)$ are the amplitudes of unsteady perturbations, k and n are the axial and azimuthal wavenumbers, respectively. The linearized amplitude equations are as follows

$$\begin{array}{rcl} -\lambda u + GrW_0iku &=& -\dfrac{dq}{dr} + Lu - \dfrac{u}{r^2} - \dfrac{2inv}{r^2}, \\ -\lambda v + GrW_0ikv &=& -\dfrac{inq}{r} + Lv - \dfrac{v}{r^2} + \dfrac{2inu}{r^2}, \\ -\lambda w + Gr(W_0^{'}u + ikwW_0) &=& -ikq + Lu + \theta, \\ -\lambda\theta + Gr(uT_0^{'} + ik\theta W_0) &=& \dfrac{1}{Pr}L\theta, \\ u^{'} + \dfrac{u}{r} + \dfrac{inv}{r} + ikw &=& 0, \end{array} \tag{13}$$

$$\tag{14}$$

where prime indicates the derivative with respect to r.

The boundary conditions at $r = 1$ are

$$u(1) = v(1) = w(1) = \theta(1) = q(1) = 0. \tag{15}$$

The boundary conditions at $r = 0$ will depend on the value of the azimuthal wave number. For $n = 0$, they are:

$$u(0) = v(0) = w^{'}(0) = \theta^{'}(0) = q(0) = 0. \tag{16}$$

For $n = 1$, they become:

$$u(0) + iv(0) = 0, \quad 2u^{'}(0) + iv^{'}(0) = 0, \quad w(0) = 0, \quad q(0) = 0. \tag{17}$$

For $n \geq 2$, they are:

$$u(0) = v(0) = w(0) = \theta(0) = q(0). \tag{18}$$

The eigenvalues of the boundary value problem (13)–(18) determine the stability of the base flow (11). The flow (11) is said to be stable if all the real parts of λ are positive, and unstable, if at least one of the real parts of λ is negative.

Numerical results and discussion

The pseudospectral collocation method based of Chebyshev polynomials is used to obtain a numerical solution to the problem (13)–(18). The functions $u(r)$, $v(r)$, $w(r)$, $q(r)$ and $\theta(r)$ are sought in the form

$$u(x) = \sum_{m=0}^{N-1} a_m T_m(x), \quad v(x) = \sum_{m=0}^{N-1} b_m T_m(x), \quad w(x) = \sum_{m=0}^{N-1} c_m T_m(x), \tag{19}$$

$$q(x) = \sum_{m=0}^{N-1} d_m T_m(x), \quad \theta(x) = \sum_{m=0}^{N-1} e_m T_m(x), \tag{20}$$

where $x = 2r - 1$, $T_m(x) = cos(m \arccos x)$ is the Chebyshev polynomial of degree m and a_m, b_m, c_m, d_m and e_m are unknown coeficients. Subsituting (19)–(20)into (13)–(18) and using

$$x_j = cos\frac{\pi j}{N-1}, \quad j = 0, 1, 2, \ldots, N-1 \tag{21}$$

as the collocation points, we obtain a generalized eigenvalue problem

$$(A - \lambda B)\mathbf{u} = 0, \tag{22}$$

where A and B are complex-valued matrices and

$$\mathbf{u} = (a_0 ... a_{N-1} b_0 ... b_{N-1} c_0 ... c_{N-1} d_0 ... d_{N-1} e_0 ... e_{N-1})^T.$$

Calculations are performed for several values of the Prandtl number. Both axisymmetric ($n = 0$) and asymmetric ($n \geq 1$) modes are analyzed. It is found that the mode with $n = 1$ is the least stable mode in accordance with linear stability theory. Calculations also show the presence of two instability modes - hydrodynamical mode and thermal mode. Similar instability patterns were obtained in [5], [6] and later in [10]–[13].

Computational results (in terms of the critical Grasshof numbers) are compared with experimental data presented in [9]. It is shown in [9] that if the Grasshof number exceeds the critical value (which depends on the Prandtl number) then the flow (11) becomes unstable. As a result of the instability a spiral vortex (with $n = 1$) was observed in all experiments. Thus, both experimental and theoretical results agree qualitatively (in terms of the structure of the critical motion). Reasonable agreement is also found between the values of the critical Grasshof numbers (for $n = 1$) and experimental data.

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