Set–valued Fenchel conjugates for vector–valued convex functions and applications to coherent risk measures

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Let X, V be locally convex spaces and $K \subseteq V$ a closed convex pointed cone generating a partial order in V. We prove that every proper closed and convex function $f : X \to V \cup \{+\infty\}$ has a set–valued affine minorant of a certain type and every such function is the pointwise supremum of its (set–valued) affine minorants. Our approach makes use of set relations in the power set of V: For $M_1, M_2 \subseteq V$, by

$$M_1 \preccurlyeq_K M_2 \quad \Longleftrightarrow \quad M_2 \subseteq M_1 + K$$

a quasiorder is introduced that replaces the usual $\leq$–relation in the real–valued case. The set–valued affine minorants have the form

$$T_{(x^*,v^*)}(x) := \{v \in V : \; x^*(x) + v^*(v) = 0\}$$

with $x^* \in X^*$ and $v^* \in K^*$, the (negative) dual cone of K. A family of conjugates f_{v^*}, $v^* \in K^* \setminus \{0\}$ is defined by

$$(-f^*_{v^*})(x^*) := \bigcup_{y \in X} \left\{ f(y) + T_{(-x^*,v^*)}(y) + K \right\}$$

where the union can be interpretated as an infimum w.r.t. $\preccurlyeq_K$. We define a (one!) biconjugate of f and present a corresponding biconjugation theorem. Finally, (natural) extensions to the case of set–valued convex functions will be discussed including a duality theory for set–valued coherent measures of risk which are a recent development in financial mathematics.