Local exact controllability of a Korteweg-de Vries equation on a critical spatial domain.

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Abstract

The boundary controllability problem for a Korteweg-de Vries equation with the Dirichlet boundary condition is considered. We study this problem for a spatial domain for which the underlying linearized control system is not controllable. We prove that the nonlinear term gives the local controllability around the origin, provided that the time of control is large enough.

1 Introduction and main results

Let $L > 0$ be fixed. Let us consider the following Korteweg-de Vries control system:

$$(1) \qquad \begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \end{cases}$$

where the state is $y(t,\cdot) : [0,L] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. It is a well known example of a nonlinear dispersive partial differential equation which has been introduced by Korteweg and de Vries in [2] to describe approximately long waves in water of relatively shallow depth.

In this talk, we are concerned with the local exact controllability of the system (1). Rosier has proved in [3] that the control system (1) is locally controllable around the origin, provided the length of the spatial domain is not critical.

Theorem 1 [3, Theorem 1.3] *Let $T > 0$ and assume that*

$$(2) \qquad L \notin N := \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}};\ k, l \in \mathbb{N}^* \right\}.$$

Then there exists $r > 0$ such that, for every $(y_0, y_T) \in L^2(0,L)^2$ with $\|y_0\|_{L^2(0,L)} < r$ and $\|y_T\|_{L^2(0,L)} < r$, there exist $u \in L^2(0,T)$ and

$$y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$$

satisfying (1) *such that $y(0,\cdot) = y_0$ and $y(T,\cdot) = y_T$.*

Moreover, Rosier proved that the underlying linearized control system of (1) around the origin is not controllable if $L \in N$. There exists a finite-dimensional subspace of $L^2(0,L)$ which is missed for the linear system.

Recently, Coron and Crépeau have proved in [1] the same theorem for the critical lengths $L = 2k\pi$ with $k \in \mathbb{N}^*$ (take $l = k$ in (2)) such that

$$(3) \qquad \nexists (m,n) \in \mathbb{N} \times \mathbb{N} \ \text{ with } \ m^2 + mn + n^2 = 3k^2 \ \text{ and } \ m \neq n.$$

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Remark 1 *The condition* (3) *has been communicated to the author by J.-M. Coron and E. Crépeau: as they pointed out, if it is not satisfied, the dimension of the missed directions subspace is higher that one and the proof given in [1] does not work anymore.*

In this work we are concerned with other critical domains. We shall consider the following set of lengths.

$$(4)\quad N' := \Big\{L \in \mathbb{R}^+ ;\, \forall m \in \mathbb{N},\, L \neq 2\pi m \text{ and } \exists!\,(k,l) \in \mathbb{N}\times\mathbb{N},\, k > l \text{ such that } L = 2\pi\sqrt{(k^2+kl+l^2)/3}\Big\}.$$

In this case, i.e. $L \in N'$, we know by the work of Rosier that the subspace of missed directions is two-dimensional. For each $L \in N'$, we shall use the notation $L = L(k,l)$ to recall the positive natural numbers k, l which define L.

We apply the same method used by Coron and Crépeau in [1] (in their case, the subspace missed is one-dimensional) to prove that the nonlinear term $y\partial_x y$ allows us to go in the directions missed. Let us explain the idea of the method. Let $y = y(t,x)$ be a solution of (1). We shall consider an expansion of (y,u) with the same scaling on the state and on the control

$$y = \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 \dots, \qquad u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 \dots$$

In this way

$$y\partial_x y = \epsilon^2 y_1\partial_x y_1 + \epsilon^3 y_1 \partial_x y_2 + \epsilon^3 y_2 \partial_x y_1 + \text{(higher terms)},$$

and for small ϵ, $y \approx \epsilon y_1 + \epsilon^2 y_2$, where y_1, y_2 satisfy

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \partial_x^3 y_1 = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ \partial_x y_1(t,L) = u_1(t), \end{cases}$$

and

$$\begin{cases} \partial_t y_2 + \partial_x y_2 + \partial_x^3 y_2 = -y_1\partial_x y_1, \\ y_2(t,0) = y_2(t,L) = 0, \\ \partial_x y_2(t,L) = u_2(t). \end{cases}$$

The strategy consists in first to prove that the expansion to the second order of $y = y(t,x)$, i.e. $\epsilon y_1 + \epsilon^2 y_2$ can go to the missed directions and then, by means of a fixed point theorem, to get the local controllability. To enable to the second order expansion of $y = y(t,x)$ to reach all the missed directions, we need a time large enough. Thus we obtain our main result:

Theorem 2 *Let $L = L(k,l) \in N'$. Let us define*

$$p := \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}}$$

Let $T > 4\pi/p$. There exists $r > 0$ such that for every $(y_0, y_T) \in L^2(0,L)^2$ with $\|y_0\|_{L^2(0,L)} < r$ and $\|y_T\|_{L^2(0,L)} < r$, there exist $u \in L^2(0,T)$ and

$$y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$$

satisfying (1) *such that $y(0,\cdot) = y_0$ and $y(T,\cdot) = y_T$.*

References

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