Inhomogeneous Random Graphs

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In the late 1950s and early 1960s Paul Erdős and Alfred Rényi wrote a series of papers in which they founded the theory of random graphs. They studied the space $\mathcal{G}(n,m)$ of random graphs $G(n,m)$ consisting of all $\binom{N}{m}$ graphs with a given set of n distinguishable vertices that have $m = m(n)$ edges, where $N = N(n) = \binom{n}{2}$. In particular, they identified thresholds for $m(n)$ that make various monotone properties likely, and used random graphs to prove the existence of graphs with paradoxical properties. It is rather trivial to carry over these results to the space $\mathcal{G}(n,p)$ of graphs with n labelled vertices, in which the edges are chosen independently, with probability $p = p(n)$. In fact, the binomial 'Erdős-Rényi' random graphs $G(n,p)$ are even easier to handle than the original graphs $G(n,m)$. For $m(n) = p(n)\binom{n}{2}$ the two models are practically indistinguishable as far as monotone properties are concerned.

These 'classical' $\mathcal{G}(n,m)$ and $\mathcal{G}(n,p)$ models are 'homogeneous' in the sense that the degrees (for example) tend to be concentrated around a typical value. Many graphs arising in the real world do not have this property, having, for example, power-law degree distributions. In recent years there has been much interest in defining and studying 'inhomogeneous' random graph models.

One of the most studied properties of these new models is their 'robustness', or, equivalently, the 'phase transition' as an edge density parameter is varied. These questions originate in the fundamental result Erdős and Rényi proved in 1961 that for $p = c/n$ the graph $G(n,p)$ undergoes a phase transition at $c = 1$: as p increases above this value, a 'giant component' appears.

Many of the new inhomogenous models are rather complicated; although there are exceptions, in most cases precise questions such as determining exactly the critical point of the phase transition are approachable only when there is independence between the edges. Fortunately, some models studied have this already, and others can be approximated by models with independence.

Recently, Svante Janson, Oliver Riordan and I introduced a very general model of an inhomogenous random graph with independence between the edges, which scales so that the number of edges is linear in the number of vertices. This scaling corresponds to the $p = c/n$ scaling for $G(n,p)$ used to study the phase transition; also, it seems to be a property of many large real-world graphs. Our model includes as special cases many models previously studied.

In the talk I shall review a number of classical results, and present some of the new results concerning this general model that I obtained jointly with Janson and Riordan, with emphasis on the phase transition, the place and type of the emergence of the giant component.