PENALIZATION OF DIRICHLET OPTIMAL CONTROL PROBLEMS *

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Abstract. We apply Robin penalization to Dirichlet optimal control problems governed by semilinear elliptic equations. Error estimates in terms of the penalization parameter are stated. Error estimates for the numerical approximation of both Dirichlet and Robin optimal control problems are provided and the optimal relation of the penalization and the discretization parameter is established. Accuracy and efficiency of both methods are compared. Finally, a numerical experiment is performed.

1. Introduction. The purpose of this paper is to study solution strategies for the control problem

$$(\mathrm{P})\left\{\begin{array}{l}\text{Min } J(u)=\displaystyle\int_\Omega L(x,y_u(x))dx+\frac{N}{2}\int_\Gamma u(x)^2d\sigma(x)\\ u\in U_{ad}=\{u\in L^2(\Gamma):\ \alpha\le u(x)\le\beta \text{ a.e. } x\in\Gamma\},\\ -\Delta y_u+a(x,y_u)=0 \text{ in } \Omega,\ y_u=u \text{ on } \Gamma,\end{array}\right.$$

where Ω is an an open convex bounded polygonal set of $\mathbb{R}^2$, Γ is its boundary, $N>0$, $-\infty<\alpha<\beta<+\infty$ and $a(x,y)$ and $L(x,y)$ satisfy natural assumptions about growth and regularity (see e.g. [5]).

For every $\varepsilon>0$, we can try to replace problem (P) by the following one

$$(\mathrm{P}_\varepsilon)\left\{\begin{array}{l}\text{Min } J_\varepsilon(u)=\displaystyle\int_\Omega L(x,y_\varepsilon(u)(x))dx+\frac{N}{2}\int_\Gamma u(x)^2d\sigma(x)\\ \alpha\le u(x)\le\beta \text{ a.e. } x\in\Gamma,\\ -\Delta y_\varepsilon(u)+a(x,y_\varepsilon(u))=0 \text{ in } \Omega,\ \varepsilon\partial_\nu y_\varepsilon(u)+y_\varepsilon(u)=u \text{ on } \Gamma.\end{array}\right.$$

We will check how solutions of these problems converge to a solution of the original one as ε goes to zero and the rate of convergence.

We refer to [6] for a modern reference in the penalty method for elliptic PDEs. In [2] the linear-quadratic control problem with no control constraints is investigated. For the numerical analysis, it is very interesting [1], where a complete study of the penalty method for the FEM is investigated. One of the conclusions of this work is that, when it is possible to solve numerically the Dirichlet problem, it is better to do so than to solve the penalized Robin problem. This is the case of P1 elements. Penalty method is then justified when it is very difficult to construct elements vanishing on the boundary of the domain. This is the case, e.g., of elements involving derivatives or of high order. In this paper we will refer to papers that investigate the approximation of both the Dirichlet and the Robin control problems with P1 elements, and we will get the same result: it is better to solve directly the Dirichlet problem.

The main result is the following one:

THEOREM 1.1. *Suppose that* $\bar{u}$ *is a local solution of* (P) *that satisfies second order sufficient optimality conditions and* $\bar{u}_\varepsilon$ *is a sequence of solutions of* (P_ε) *converging*

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to $\bar{u}$. Then $\bar{u}_\varepsilon$ is uniformly bounded in $H^{1-1/p}(\Gamma)$,there exists a subsequence of $\bar{u}_\varepsilon$, still indexed in the same way, converging uniformly to $\bar{u}$ and there exists $C > 0$ and $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$

$$\|\bar{u} - \bar{u}_\varepsilon\|_{L^2(\Gamma)} \leq C\varepsilon^{1-1/p}.$$

2. Applications to the numerical analysis. In [5] it is proved that for a P_1 discretization by the FEM for problem (P), one gets

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq Ch^{1-1/p}. \tag{2.1}$$

From [3], we can deduce that for fixed ε, the same type of discretization for (P_ε) leads to

$$\|\bar{u}_\varepsilon - \bar{u}_{\varepsilon,h}\|_{L^2(\Gamma)} \leq C\frac{h^{3/2}}{\varepsilon}, \tag{2.2}$$

if the number of points in the boundary of $\Gamma_s = \{x \in \Gamma : \bar{u}_\varepsilon(x) = \alpha \text{ or } \bar{u}_\varepsilon(x) = \beta\}$ in the topology of Γ is finite, which is an assumption not easy to verify *a priori* but that is fulfilled in most practical cases.

From the result of Theorem 1.1 and inequality (2.2), we obtain that

$$\|\bar{u} - \bar{u}_{\varepsilon,h}\|_{L^2(\Gamma)} \leq C\varepsilon^{1-1/p} + C\frac{h^{3/2}}{\varepsilon}.$$

The optimal election for $\varepsilon = h^\lambda$ is then for $\lambda = 3p/(4p-2)$, which leads to

$$\|\bar{u} - \bar{u}_{\varepsilon,h}\|_{L^2(\Gamma)} \leq Ch^{(3p-3)/(4p-2)}.$$

It is clear that for $p > 2$

$$\frac{3p-3}{4p-2} < 1 - \frac{1}{p},$$

and hence, when we are using P_1 elements, it seems better to solve Dirichlet boundary control problems by the method proposed in [5] than by the penalty method, even if the assumption on Γ_s is satisfied.

In [4] the approximation of the controls is performed by piecewise constant discrete controls. In this case $\varepsilon\|\bar{u}_\varepsilon - \bar{u}_{\varepsilon,h}\|_{L^2(\Gamma)} = O(h)$, the optimal λ is $\lambda = p/(2p-1)$ and the order of convergence is $(p-1)/(2p-1) < 1 - 1/p$. The computational effort for the optimization is the same, since the number of variables is the same.

REFERENCES

[1] I. Babuška, *The finite element method with penalty*, Math. Comp., 27 (1973), pp. 221–228.

[2] F. Ben Belgacem, H. El Fekih, and H. Metoui, *Singular perturbation for the dirichlet boundary control of elliptic problems*, ESAIM: M2AN, 37 (2003), pp. 833–850.

[3] E. Casas and M. Mateos, *Error estimates for the numerical approximation of neumann control problems.*, COAP, (2006). Submitted.

[4] E. Casas, M. Mateos, and F. Tröltzsch, *Error estimates for the numerical approximation of boundary semilinear elliptic control problems*, Computational Optimization and Applications, 31 (2005), pp. 193–219.

[5] E. Casas and J.-P. Raymond, *Error estimates for the numerical approximation of dirichlet boundary control for semilinear elliptic equations*, (To appear).

[6] M. Costabel and M. Dauge, *A singularly perturbed mixed boundary value problem*, Commun. Partial Differential Equations, 21 (1996), pp. 1919–1949.